The Conformal Generators on the Fuzzy Sphere

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Based on 2409.02998 with Liam Fitzpatrick and Ami Katz

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Introduction and motivations

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Why? We are interested in High energy states

- Explore explicitly the chaotic behavior
- Study deformations of the CFT using Hamiltonian truncation

$$H = H_{\mathrm{CFT}} + \lambda \int d^{d-1}x\, \mathcal{O}$$

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Today's Goal: Select "good" CFT states by constructing **conformal generators** on the fuzzy sphere.

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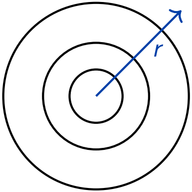
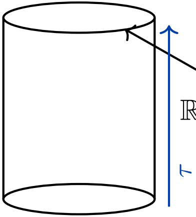
General setup

$\mathbb{R}^d$



r	$\longleftrightarrow$	e^{τ}
CFT ops		$\mathcal{H}_{S^{d-1}}$
Δ		E



$\mathbb{R} \times S^{d-1}$

General setup

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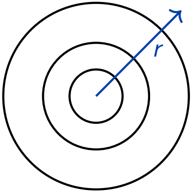

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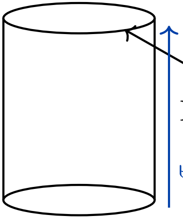

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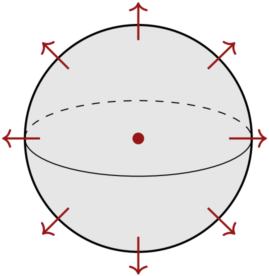
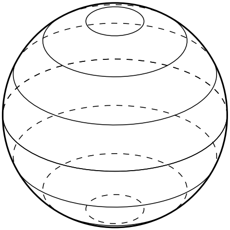


[Haldane (83), Greiter (11)]

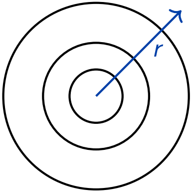
LLL projection $\Longrightarrow$

$(2s+1)$-fold degenerate



$A = s \cos\theta d\phi \quad \int dA = 4\pi s$

General setup



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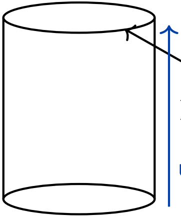
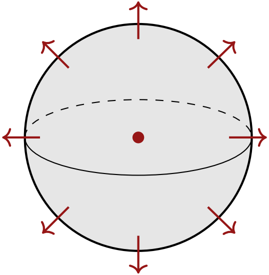
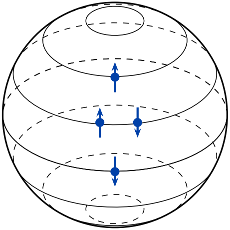




$$\xRightarrow{\text{LLL projection}}$$

$N_e = 2s + 1$

Half filling



Advantage: Preserve perfect rotational invariance

3d Ising CFT

Non-relativistic fermions on S^2 in the presence of the magnetic field $|B| = s/R^2$. $H \equiv R^2 \int d\Omega \mathcal{H}$

$$\mathcal{H}_{\text{free}} \sim \psi^\dagger(\Omega)(\partial_\mu + iA_\mu)^2 \psi(\Omega) \xrightarrow{LLL} \psi(\Omega) = \frac{1}{R} \sum_{m=-s}^{s} \Phi_m(\Omega)(c_{m,\uparrow}, c_{m,\downarrow})^T$$

$$\Phi_m(\Omega) = \mathcal{N}_m e^{im\phi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right) \qquad \text{Monopole harmonics}$$

$$\{\psi^\dagger(\Omega), \psi(\Omega')\} = \sum_{m=-s}^{s} \frac{\Phi_m^*(\Omega)\Phi_m(\Omega')}{R^2} = \frac{2s+1}{4\pi R^2} \cos^{2s} \frac{\delta\theta}{2}, \quad \delta\theta_{\text{UV}} \sim \frac{1}{\sqrt{s}}$$

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$$\mathcal{H} = \sum_n \left(\lambda_n (\psi^\dagger\psi) \frac{\nabla^{2n}_{S^2_1}}{R^{2n}} (\psi^\dagger\psi) - \lambda_{n,z} (\psi^\dagger\sigma^z\psi) \frac{\nabla^{2n}_{S^2_1}}{R^{2n}} (\psi^\dagger\sigma^z\psi) \right) - h\psi^\dagger\sigma^x\psi$$

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Ising phase transition

$$h \gg 0\,, \qquad |\psi_x\rangle = \prod_m |+\hat{x}\rangle_m$$

$$h = 0\,, \qquad |\psi_\pm\rangle = \prod_m |\pm\hat{z}\rangle_m$$

Ising symmetries

- SO(3) invariance
- $\mathbb{Z}_2$ symmetry: $\uparrow \leftrightarrow \downarrow$
- Parity: particle $\leftrightarrow$ hole

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Generators from $T^{\mu\nu}$

In a CFT, Noether currents for the conformal symmetries $x^\mu \to x^\mu + \epsilon^\mu(x)$ can be written in terms of the energy-momentum tensor

$$j_\epsilon^\mu(x) = \epsilon^\nu(x) T^\mu_{\ \nu}(x) \quad \Rightarrow \quad Q_\epsilon = \int d^{d-1}x \sqrt{g} j_\epsilon^0(x)$$

In $\mathbb{R} \times S^2$ ($A = 1, 2, 3$ are embedding coordinates for $S^2 \subset \mathbb{R}^3$)

$$D = \int d^2\Omega\, T^0_{\ 0} \qquad J^B \propto \int d^2\Omega \epsilon^{ABC} \hat{x}^C T^{0A}$$

$$P^A = \int d^2\Omega (\hat{x}^A T^0_{\ 0} + iT^{0A}) \qquad K^A = \int d^2\Omega (\hat{x}^A T^0_{\ 0} - iT^{0A})$$

P+K depends **only** on $T^0_{\ 0}$!

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Unfortunately the fuzzy sphere does not give us access to the CFT $T^\mu_{\ \nu}$ (conformal only in the IR) and the emergent stress tensor in the CFT does not correspond to a simple local operator in the microscopic description.

Generators from $\mathcal{H}$: Dilatation

Construct $T^0{}_0$ from the Hamiltonian density $\mathcal{H}$ and tune to criticality

$$H = \gamma H_{\mathrm{CFT}} + \sum_{\mathcal{O}\ \mathrm{primary}} g_{\mathcal{O}} \int d^2\Omega\, \mathcal{O}(\Omega) \qquad g_{\mathcal{O}} \sim \left(\frac{1}{\sqrt{s}}\right)^{\Delta_{\mathcal{O}}-3}$$

Remove contributions from $\epsilon(x)\,(\Delta \sim 1.4)$ and $\epsilon'(x)\,(\Delta \sim 3.8)$ tuning λ_n, h using Conformal Perturbation Theory [Lao, Rychkov (23)].

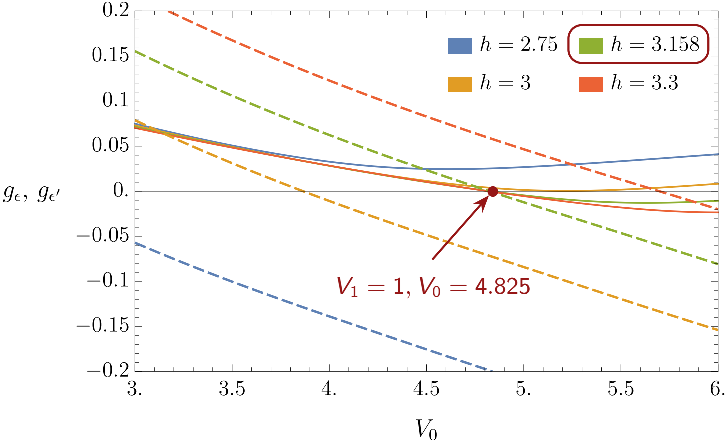
$$H\,|\mathcal{O}_i\rangle = E_i\,|\mathcal{O}_i\rangle \qquad\qquad E_i = \gamma\Delta_i + g_{\mathcal{O}}\delta E_i^{(\mathcal{O})}$$

$$\delta E_i^{(\mathcal{O})} = \frac{\langle\mathcal{O}_i|\int d^2\Omega\mathcal{O}\,|\mathcal{O}_i\rangle}{\langle\mathcal{O}_i|\mathcal{O}_i\rangle} = \begin{cases} f_{\mathcal{O}_i\mathcal{O}\mathcal{O}_i} & \text{primary} \\ f_{\mathcal{O}_i\mathcal{O}\mathcal{O}_i}\left(1 + \frac{\Delta_{\mathcal{O}}(\Delta_{\mathcal{O}}-3)}{6\Delta_{\mathcal{O}_i}}\right) & \partial\mathcal{O}_i \end{cases}$$

$$\min_{\gamma, g_\epsilon, g_{\epsilon'}} \sum_{\mathcal{O}_i} \left(E_i - \gamma\Delta_i - \delta E_i^{(\epsilon)} - \delta E_i^{(\epsilon')}\right)^2$$

Generators from $\mathcal{H}$: Dilatation

$\mathcal{O}_i = \epsilon,\, \sigma,\, \epsilon',\, \partial\epsilon\,\partial\sigma$ $\qquad$ $N = 2s + 1 = 12$



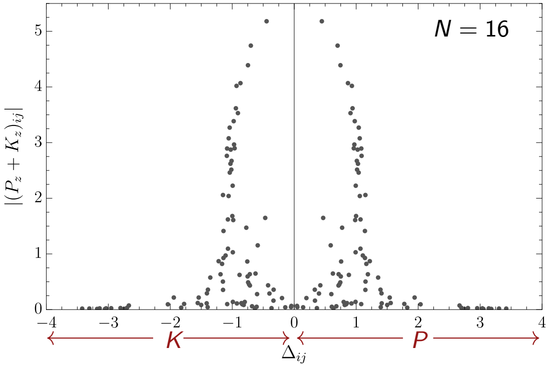
Haldane pseudopentials V_0 and V_1 are related to the couplings λ_0 and λ_1

Δ_i and $f_{\mathcal{O}_i\mathcal{O}\mathcal{O}_i}$ fixed by numerical bootstrap results

Generators from $\mathcal{H}$: Translations and SCT

$$\boxed{\Lambda^A = P^A + K^A = 2\int d^2\Omega \hat{x}^A T^0_0}$$

- T^{0A} naively constructed from the microscopic charge density for rotations is not useful to construct translations and SCT $\Rightarrow$ build Λ^A
- We can still obtain $P^A - K^A = [D, P^A + K^A]$. In practice, we define P and K by selecting matrix elements such that $\Delta \to \Delta \pm 1$



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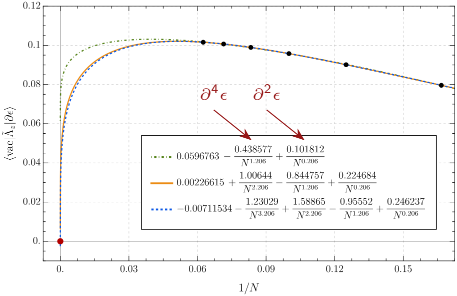
- $\tilde{\Lambda}^A = \int d^2\Omega \hat{x}^A \mathcal{H}$ with $\mathcal{H}$ tuned as before, but still not good enough because of contributions from descendant operators



$$\tilde{\Lambda}_A \supset 2\int x_A \sum_{n=0}^{\infty} g_{\nabla^{2n}\mathcal{O}} \nabla^{2n}\mathcal{O}$$

$$= 2\left[\sum_{n=0}^{\infty}(-2)^n g_{\nabla^{2n}\mathcal{O}}\right]\int x_A \mathcal{O}$$

Generators from $\mathcal{H}$: Translations and SCT

Idea: tune away descendant operators

$$\tilde{\Lambda}_A = \gamma_{\mathrm{eff}} \Lambda_A + \sum_{\mathcal{O}\ \text{primary}} g_{\mathcal{O}}^{\mathrm{eff}} \int d^2\Omega\, x_A \mathcal{O}$$

Keeping the states fixed, we use V_0 and h to tune the generators

$$\boxed{\langle \mathrm{vac}| \tilde{\Lambda}_z |\partial_A \epsilon\rangle = 0 \qquad \langle \mathrm{vac}| \tilde{\Lambda}_z |\partial_A \partial^2 \epsilon\rangle = 0 \qquad \langle \epsilon | \tilde{\Lambda}_z |\partial_A \epsilon\rangle = \sqrt{2\Delta_\epsilon}}$$

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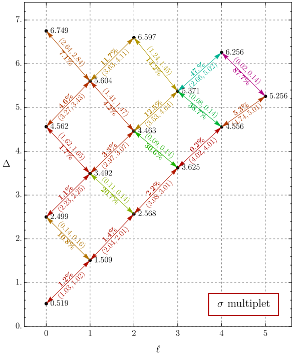

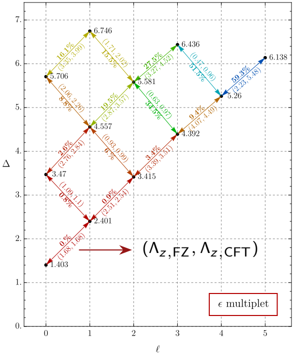

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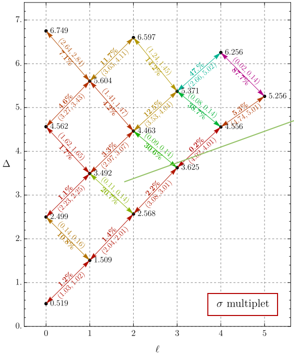
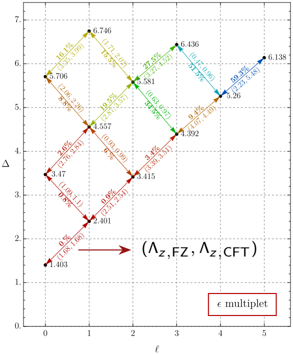


Almost null

$|P^A | \nabla_A \sigma \rangle|^2 \propto \Delta_\sigma - \frac{1}{2}$



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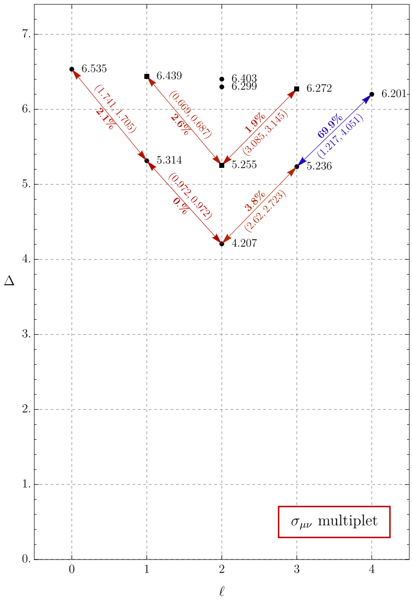

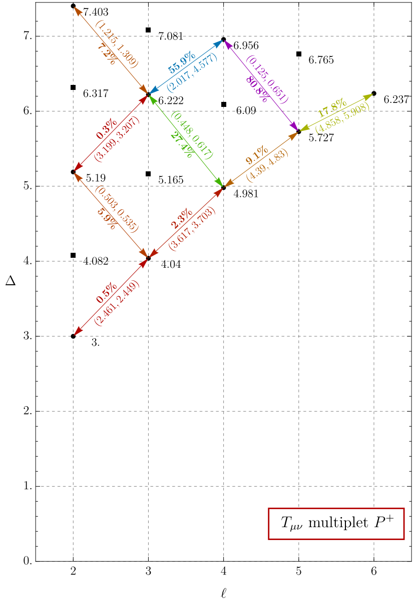

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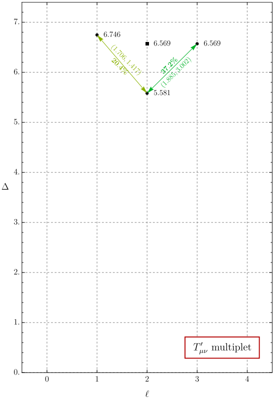
Λ_z matrix elements

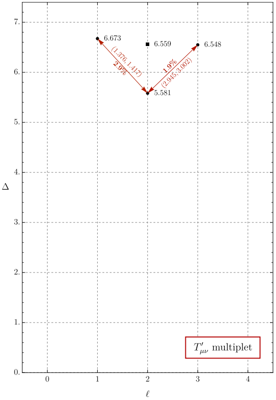
As the energy increases spectrum becomes really dense. We observe that H-eigenstates are **no longer good** K-eigenstate $\Rightarrow$ Prioritize K!

$$K_A \left|\mathcal{O}\right\rangle \approx 0 \qquad\qquad \left|\nabla_A \mathcal{O}\right\rangle = P_A \left|\mathcal{O}\right\rangle$$

New states are linear combinations of previous H-eigenstates.

H-eigenstates





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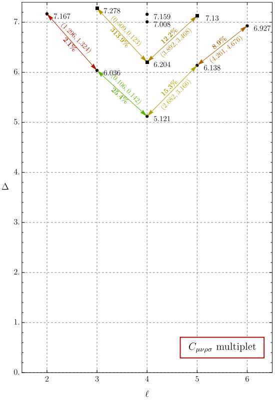
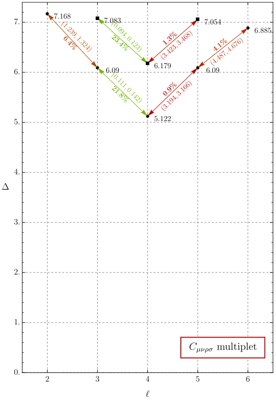
H-eigenstates





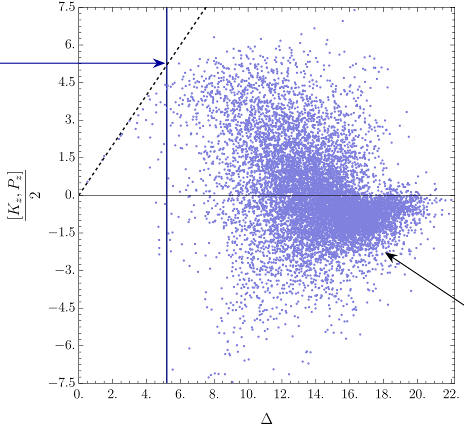
K-eigenstates

Commutator

To test the generators just constructed we can check how accurately they satisfy the conformal algebra

$$\langle\psi|\,[K_z, P_z]\,|\psi\rangle \stackrel{?}{=} 2\,\langle\psi|\,H\,|\psi\rangle$$

Rough cutoff estimate $\Lambda^{\mathrm{UV}} \approx 5.25$



All H-eigenvalues @ N=10

$\mathrm{tr}([K_z, P_z]) = 0$

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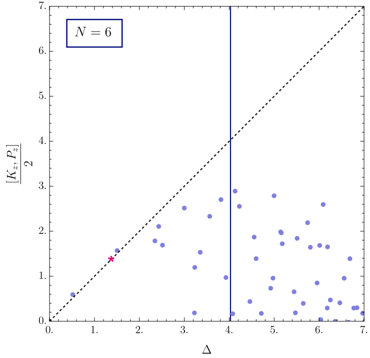
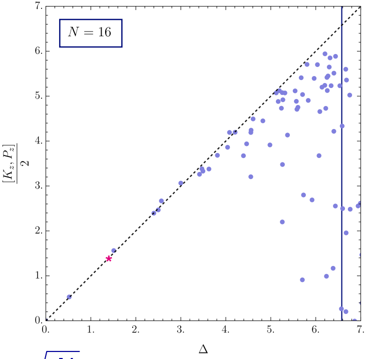
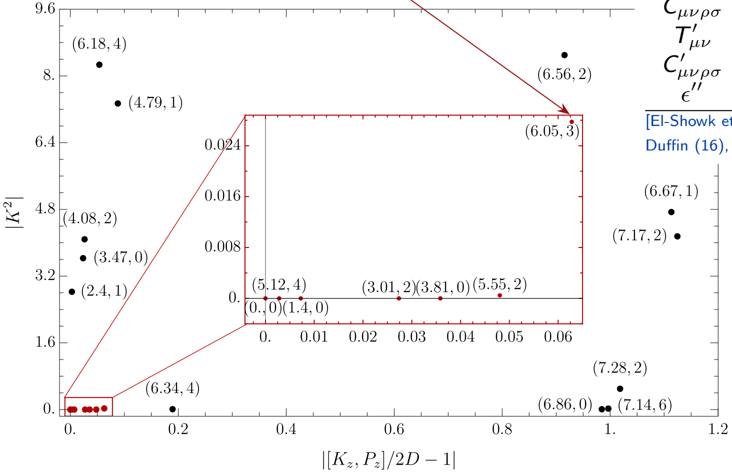




$$\Lambda_N^{\mathrm{UV}} \equiv \Lambda_{10}^{\mathrm{UV}}\sqrt{\frac{N}{10}}$$

Constructing primaries

It has to be small!
Descendant close to unitary bound



$\mathbb{Z}_2$-even

$\mathcal{O}$	Δ
ϵ	1.412625(10)
$T_{\mu\nu}$	3
ϵ'	3.82951(61)
$C_{\mu\nu\rho\sigma}$	5.022665(28)
$T'_{\mu\nu}$	5.50915(44)
$C'_{\mu\nu\rho\sigma}$	6.42065(64)
ϵ''	6.8956(43)

[El-Showk et al (12), Simmons-Duffin (16), Reehorst (21)]

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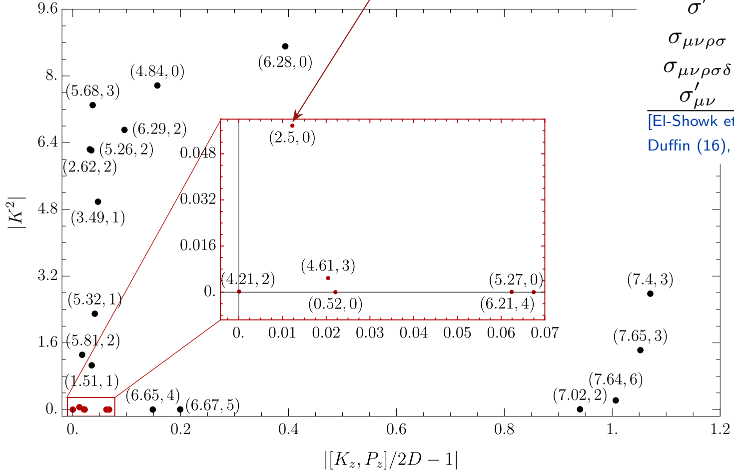


$\mathbb{Z}_2$-odd

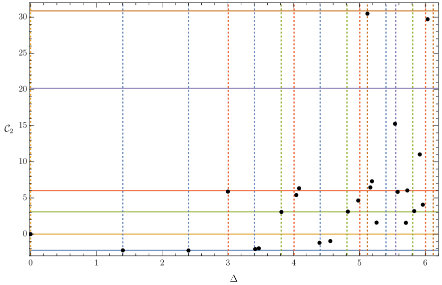
$\mathcal{O}$	Δ
σ	0.5181489(10)
$\sigma_{\mu\nu}$	4.180305(18)
$\sigma_{\mu\nu\rho}$	4.63804(88)
σ'	5.262(89)
$\sigma_{\mu\nu\rho\sigma}$	6.112674(19)
$\sigma_{\mu\nu\rho\sigma\delta}$	6.709778(27)
$\sigma'_{\mu\nu}$	6.9873(53)

[El-Showk et al (12), Simmons-Duffin (16), Reehorst (21)]

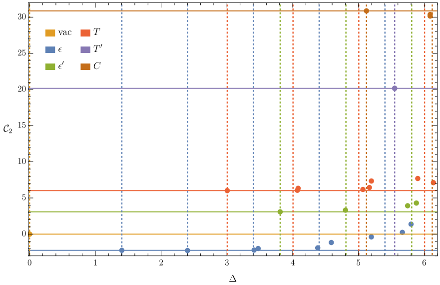
Conformal Casimir

$$\boxed{\mathcal{C}_2 = D^2 + J^2 - \frac{1}{2}\{K_A, P_A\} = D(D-3) + \ell(\ell+1) - P_A K_A}$$

Energy eigenstates



Conformal multiplets w/ K, P



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- Introduce new parameters and tune even better
- Go to higher N. Maybe by restricting to a specific spin sector?
- Compute OPE coefficients [Hu, He, Zhu (23)]. Use new OPE data for Hamiltonian truncation ($3d$ Ising Field Theory)

Thank you!