Verlinde's formula in logarithmic conformal field theory

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Two-dimensional conformal field theory

A conformal field theory (CFT) is a quantum field theory that is invariant under conformal transformations. In two dimensions, there is an infinite-dimensional algebra of local conformal transformations,

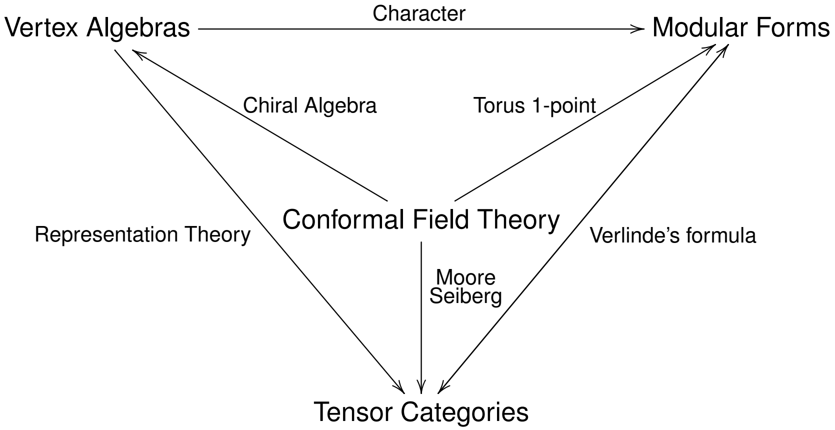
$$\mathrm{Witt} \oplus \overline{\mathrm{Witt}} = \left\{ -z^n \frac{d}{dz}, -\bar{z}^n \frac{d}{d\bar{z}} \,\middle|\, \quad n \in \mathbb{Z} \right\}$$

and much allows for a good mathematical treatment.

The Witt algebra admits a central extension to the Virasoro algebra

$$0 \to \mathbb{C} \to \mathrm{Vir} \to \mathrm{Witt} \to 0.$$

Two-dimensional conformal field theory



Vertex Operator Algebra (VOA)

A vertex operator (super)algebra $(V, |0\rangle, T, Y)$ is:

- a vector(super) space V (usually over $\mathbb{C}$).
- fields
$$Y(\ \ , z) : V \to \mathrm{End}(V)[[z^{\pm 1}]].$$
- a vacuum $|0\rangle \in V$.
- a translation operator $T \in \mathrm{End}(V)$.
- an action of the Virasoro algebra on V.
- Satisfying various axioms, most importantly locality
$$(z - w)^N [Y(v, z), Y(v', w)] = 0,$$
for all $v, v' \in V$ and sufficiently large $N \in \mathbb{Z}$.
- One writes V for $(V, |0\rangle, T, Y)$.

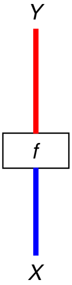
Vertex Operator Algebra (VOA)

- There is a similar notion for modules. These are typically infinite dimensional vector spaces
- Fields in the physics sense are intertwining operators that realize a tensor product on the category of modules provided that suitable finiteness conditions hold.
- The action of intertwining operator is not commutative or associative, but only commutative and associative up to natural isomorphisms that come from analytic properties of the intertwining operators. This means that the tensor category is braided and should be a ribbon category.

Ribbon Categories

A ribbon category is a balanced braided tensor category with duality. In such a category one likes to depict morphisms by diagrams, e.g.

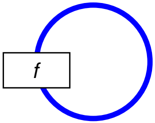
$$f : X \to Y \quad =$$



Ribbon Categories

Duality implies that one can take the trace of endomorphisms, pictorially

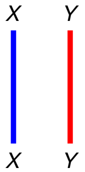
$$\mathrm{tr}_X(f : X \to X) \quad =$$



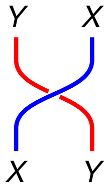
Ribbon Categories

The diagrams for tensor product and braiding are

$$X \otimes Y =$$



$$c_{X \otimes Y} : X \otimes Y \to Y \otimes X =$$



Modular Group

The modular group

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

acts on the upper half $\mathbb{H}$ of the complex plane $\mathbb{C}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .\tau = \frac{a\tau + b}{c\tau + d}$$

Modular Function

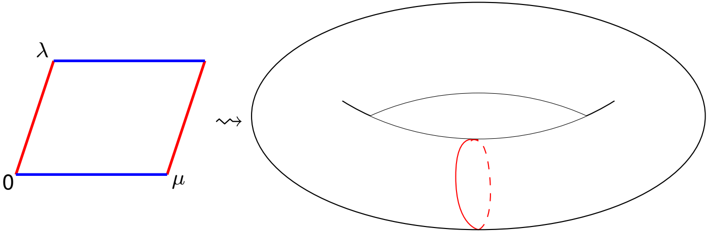
Let $\rho : SL(2, \mathbb{Z}) \to \text{End}(V)$ be a representation of $SL(2, \mathbb{Z})$. A function $f : V \times \mathbb{H} \to \mathbb{C}$ is called a vector-valued modular function if

$$f\left(v, \frac{a\tau + b}{c\tau + d}\right) = f\left(\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v, \tau\right)$$

and if $f(v, \tau)$ is meromorphic at ∞.

Tori

A complex torus can be obtained by gluing a parallelogram

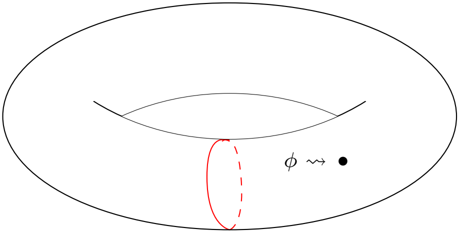


It is parameterized by $(\lambda, \mu) \in \mathbb{C}^2$ and the corresponding torus is $\mathbb{C}/\mathbb{Z}\lambda + \mathbb{Z}\mu$. Two tori are called equivalent if they are conformally homeomorphic and the moduli space of tori up to equivalence is $\mathbb{H}/SL(2,\mathbb{Z})$ with representatives $T_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$

Torus one-point functions

In conformal field theory the quantities of interest are correlation functions. A simple example is a torus one-point function, where one inserts a single field. If this field is the identity, and if the CFT is rational, then the corresponding characters form a vector-valued modular function and for general fields a vector-valued modular form.

$\langle \phi(z) \rangle_{T_\tau} \rightsquigarrow$



Axiomatics of rational conformal field theory

- In the late 1980's, Verlinde conjectured a connection between the tensor product rules of modules and their modular S-transformation, $S : \tau \mapsto -\frac{1}{\tau}$.
- This conjecture follows from the axiomatics of conformal field theory of Moore-Seiberg.
- These axiomatics say that a consistent system of correlation functions determines a modular tensor category.

Strongly rational vertex operator algebras

- The vertex algebra associated to a rational CFT is called strongly rational. Yi-Zhi Huang, 2005, showed that the correlation functions of modules of a strongly rational VOA indeed satisfies the proposed equations of Moore and Seiberg.
- This should be interpreted as strongly rational VOAs indeed describe rational CFTs, i.e. the notion of strongly rational VOAs is justified.
- The high-light of Huang's work is the proof of Verlinde's conjecture.
- Most VOAs are non-rational. Let's also focus on a possible Verlinde formula.

Modularity and Verlinde's formula

- Let $\mathcal{C}$ be a tensor category with flat tensor product.
- Let $K(\mathcal{C})$ be its Grothendieck ring and $[X]$ the class of an object X.
- For X, Y objects define the fusion rules $N_{X,Y}^{\,Z}$ via

$$[X][Y] = \sum_{Z} N_{X,Y}^{\,Z}\, [Z]$$

  where the sum is over inequivalent simple objects in $\mathcal{C}$. That is

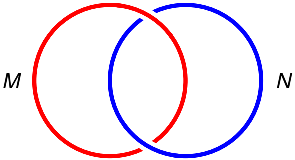
$$N_{X,Y}^{\,Z} = [X \otimes Y : Z]$$

- One is interested in these fusion rules (and the actual tensor product multiplicities).

Modularity and Verlinde's formula

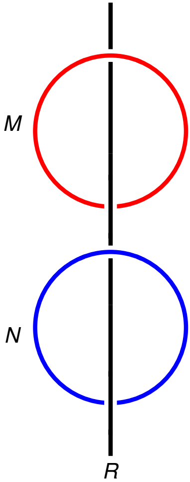
- A modular tensor category is a finite, semi-simple ribbon category with trivial Müger center.
- In a modular tensor category the modular group acts on the vector space indexed by inequivalent simple objects and the modular S-transformation corresponds up to normalization to the Hopf link

$$S_{M,N} =$$

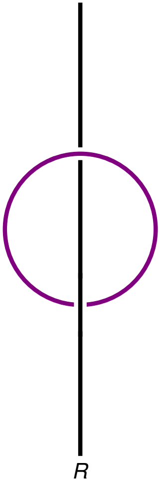


Modularity and Verlinde's formula

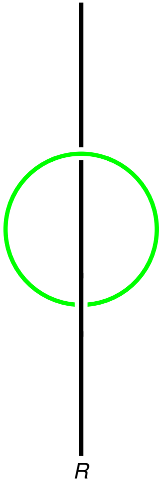
Open Hopf links satisfy



$= M \otimes N$



$= \sum_T N_{M,N}{}^T \; T$



Modularity and Verlinde's formula

Taking the trace of this identity gives (**1** the tensor identity)

$$\frac{S_{M,R}}{S_{\mathbf{1},R}} \frac{S_{N,R}}{S_{\mathbf{1},R}} = \sum_T N_{M,N}^{\;\;T} \frac{S_{T,R}}{S_{\mathbf{1},R}}$$

From this one sees that the quantum dimensions,

$$q_M : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}, \qquad R \mapsto \frac{S_{M,R}}{S_{\mathbf{1},R}}$$

form a representation of the Grothendieck ring $K(\mathcal{C})$.
If the S-matrix is invertible then

$$N_{M,N}^{\;\;T} = \sum_R \frac{S_{M,R} S_{N,R} S^{-1}_{R,T}}{S_{\mathbf{1},R}}$$

Modularity and Verlinde's formula

- Ordinary modules are graded by conformal weight with finite dimensional weight spaces, this leads to characters

$$\mathrm{ch}[M] = \sum_n \dim(M_n) q^n$$

- For a strongly rational VOA and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathrm{ch}[M](-1/\tau) = \sum_N S_{M,N}\, \mathrm{ch}[N](\tau)$$

  where the sum is over inequivalent simple modules. (Zhu)
- In a strongly rational VOA the categorical modular group action and the one on characters agree and hence Verlinde's conjecture is true (Huang, 2005)

Virasoro minimal models

- Let $M(u, v)$ be the simple Virasoro vertex algebra at central charge
$$c^{\mathrm{Vir}} = 1 - 6\frac{(u-v)^2}{uv}.$$
- $M(u, v)$ is strongly rational if $u, v \in \mathbb{Z}_{\geq 2}$.
- Its simple modules are denoted $M_{r,s}$ with $1 \leq r \leq u-1$, $1 \leq s \leq v-1$ and $M_{r,s} \cong M_{u-r,v-s}$.
- Their top level has conformal weight
$$\Delta^{\mathrm{Vir}}_{r,s} = \frac{(r-ts)^2 - (1-t)^2}{4t}, \qquad t = \frac{u}{v}.$$

Virasoro minimal models: modularity

- Their characters are, $q = e^{2\pi i\tau}$

$$\chi_{r,s}(\tau) = \frac{q^{\Delta^{\mathrm{Vir}}_{r,s} - c^{\mathrm{Vir}}/24 + 1/24}}{\eta(q)} \sum_{j\in\mathbb{Z}} \left[q^{j(uvj+vr-us)} - q^{(uj-r)(vj-s)} \right]$$

- Let $I = \{(r,s) | 1 \leq r \leq u-1, 1 \leq s \leq v-1\}/\sim$ with $(r,s) \sim (u-r, v-s)$.
- Modular S-transformation is

$$\chi_{r,s}(-1/\tau) = \sum_{(r',s')\in I} \mathrm{S}_{(r,s)(r',s')} \chi_{r',s'}$$

with modular S-matrix entries

$$\mathrm{S}_{(r,s)(r',s')} = -2\sqrt{\frac{2}{uv}} (-1)^{rs'+r's} \sin\frac{v\pi rr'}{u} \sin\frac{u\pi ss'}{v}$$

Virasoro minimal models: Verlinde's formula

- Verlinde fusion rules

$$N_{(r,s)(r',s')}{}^{(r'',s'')} := \sum_{(R,S)\in I} \frac{\mathrm{S}_{(r,s)(R,S)}\mathrm{S}_{(r',s')(R,S)}\mathrm{S}^{*}_{(r'',s'')(R,S)}}{\mathrm{S}_{(1,1)(R,S)}}$$

- Evaluates to

$$N_{(r,s)(r',s')}{}^{(r'',s'')} = N^{u}_{r,r'}{}^{r''} N^{v}_{s,s'}{}^{s''}$$

with

$$N^{w}_{t,t'}{}^{t''} = \begin{cases} 1 & \text{if } |t-t'|+1 \leqslant t'' \leqslant \min\{t+t'-1, 2w-t-t'-1\} \\ & \text{and if } t+t'+t'' \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Virasoro minimal models: Verlinde's formula

Indeed

$$M_{r,s} \boxtimes M_{r',s'} \cong \bigoplus_{(r'',s'')\in I} N_{(r,s)(r',s')}{}^{(r'',s'')} \, M_{r'',s''}$$

Modularity and Verlinde's formula

- A general hope is that this holds in some way for much more classes of VOAs.
- We are interested in non semisimple and non finite categories.
- The open Hopf link argument is fine, a difficult technical challenge is however rigidity.
- The S-matrix will then be replaced by a S-kernel.
- The VOA situation however seems unclear.
- There are VOAs associated to Lie algebras and we will illustrate the problem in the example of $\mathfrak{g} = \mathfrak{sl}_2$.

Affine VOAs (WZW theories)

- $\mathfrak{g}$ Lie algebra with invariant symmetric bilinear form κ.
- B be a basis of $\mathfrak{g}$.
- The affinization $\widehat{\mathfrak{g}}$ has basis $\{K, d, x_n | n \in \mathbb{Z}, x \in B\}$ with d a derivation and K central and relations

$$[x_n, y_m] = [x, y]_{n+m} + K\delta_{n+m,0}\kappa(x, y).$$

- A $\mathfrak{g}$-module M lifts to a $\widehat{\mathfrak{g}}$-module at level $k \in \mathbb{C}$, via first lifting to a $\widehat{\mathfrak{g}}_{\geq 0}$-module via $x_n M = 0$ for $n > 0$ and $K = k\mathrm{Id}_M$. Then

$$V^k(M) := \mathrm{Ind}_{\widehat{g}_{\geq 0}}^{\widehat{g}} M$$

- $L_k(M)$ its almost semisimple quotient.
- $V^k(\mathfrak{g}) = V^k(\mathbb{C})$ affine VOA and $L_k(\mathfrak{g})$ is its unique graded simple quotient.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$

- standard basis $\{e, f, h\}$ with

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h$$

  and $\kappa(e, f) = 1, \kappa(h, h) = 2$.
- Casimir

$$C = \frac{1}{2}hh + ef + fe.$$

- finite dimensional modules: e.g. R_n the $n + 1$-dimensional simple module of highest-weight n.
- infinite-dimensional highest-weight modules.
- weight modules $E_{\lambda,\Delta}$ that are neither highest nor lowest weight.

A weight module of $\mathfrak{g} = \mathfrak{sl}_2$

$$E_{\lambda,\Delta} = \bigoplus_{\mu \in 2\mathbb{Z}} \mathbb{C} v_{\lambda+\mu}$$

$$Cv_{\lambda+\mu} = \Delta v_{\lambda+\mu},$$

$$hv_{\lambda+\mu} = (\lambda+\mu) v_{\lambda+\mu},$$

$$ev_{\lambda+\mu} = v_{\lambda+\mu+2}$$

$$fv_{\lambda+\mu} = fev_{\lambda+\mu-2} = \frac{1}{4}(2C - hh - 2h)\, v_{\lambda+\mu-2}$$

$$= \frac{1}{4}(2\Delta - (\lambda+\mu-2)(\lambda+\mu))\, v_{\lambda+\mu-2}$$

A weight module of $\mathfrak{g} = \mathfrak{sl}_2$

- The tensor product of two such weight modules is not a finite direct sum of weight modules.
- $$fv_{\lambda+\mu} = \frac{1}{4}\left(2\Delta - (\lambda+\mu-2)(\lambda+\mu)\right) v_{\lambda+\mu-2}$$

  hence $E_{\lambda,\Delta}$ is simple if $\Delta \neq \frac{(\lambda+\mu)(\lambda+\mu-2)}{2}$ and otherwise

  $$0 \to D^-_{\lambda+\mu} \to E_{\lambda,\Delta} \to D^+_{\lambda+\mu-2} \to 0$$

  with lowest-weight submodule $D^-_{\lambda+\mu}$ and highest-weight module $D^+_{\lambda+\mu-2}$ as quotient.

- k admissible

$$k+2=t=\frac{u}{v}, \qquad u\in\mathbb{Z}_{\geq 2}, \quad v\in\mathbb{Z}_{\geq 1}, \quad \gcd\{u,v\}=1.$$

- ω the fundamental weight

$$\lambda_{r,s}=(r-1-ts)\omega.$$

- The admissible conformal weights

$$\Delta_{r,s}=\frac{(vr-us)^2-v^2}{4uv}$$

- short-hand notation

$$\mathcal{D}^{\pm}_{r,s}:=L_k(D^{\pm}_{\pm\lambda_{r,s}}), \quad \mathcal{L}_r:=L_k(R_r), \quad \mathcal{E}_{\lambda;\Delta_{r,s}}:=L_k(E_{\lambda,\Delta_{r,s}})$$

Theorem (Adamović-Milas 1996)

Let $k = -2 + \frac{u}{v}$ be an admissible level. Then, the simple objects in $L_k(\mathfrak{sl}_2)\text{-wtmod}_{\geq 0}$ are exhausted, up to isomorphism, by the following list:

Atypical:

- *The $\mathcal{L}_r$, for $r = 1, \ldots, u-1$;*
- *The $\mathcal{D}^{\pm}_{r,s}$, for $r = 1, \ldots, u-1$ and $s = 1, \ldots, v-1$;*

Typical:

- *The $\mathcal{E}_{\lambda;\Delta_{r,s}}$, for $r = 1, \ldots, u-1$, $s = 1, \ldots, v-1$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ with $\lambda \neq \lambda_{r,s}, \lambda_{u-r,v-s} \pmod{2\mathbb{Z}}$.*

The modules in this list are all mutually non-isomorphic.

Physical sickness

- Given the success of integer level WZW theories, physcis was naturally interested in the fractional level case.
- in the 1990's one was only aware of the atypical modules and thanks to Kac-Wakimoto it seemed as the characters of those have good modular properties, i.e. they seemed to be vector-valued meromorphic Jacobi forms.
- Application of Verlinde's formula gave integer fusion coefficients that were sometimes negative.
- A negative integer is not a dimension and so the fractional WZW theories were discarded as physical sick.
- Their misconception was that characters are formal power series that agree with the expansion of meromorphic Jacobi forms in a suitable domain, but the modular (Jacobi) group does not preserve such domains.

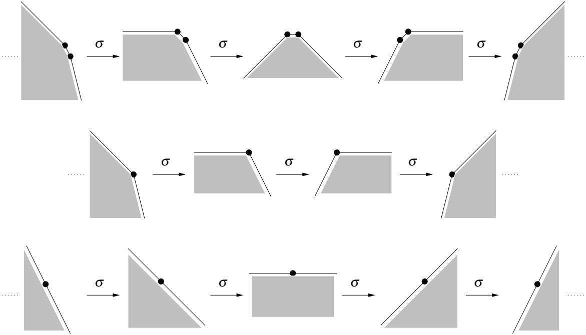
Some modules in $L_k(\mathfrak{sl}_2)$-wtmod

- $$[h_m, e_n] = 2e_{m+n}, \quad [h_m, h_n] = 2m\delta_{m+n,0}K,$$
  $$[e_m, f_n] = h_{m+n} + m\delta_{m+n,0}K, \quad [h_m, f_n] = -2f_{m+n}.$$

- spectral flow automorphism for $\ell \in \mathbb{Z}$,
  $$\sigma^\ell(e_n) = e_{n-\ell}, \qquad \sigma^\ell(f_n) = f_{n+\ell}, \qquad \sigma^\ell(h_n) = h_n - \delta_{n,0}\ell K$$
  $$\sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 K$$

- M in $L_k(\mathfrak{sl}_2)$-wtmod, then $\sigma_\ell^*(M)$ isomorphic to M as vector space but $x \in \widehat{\mathfrak{sl}_2}$ acts by $\sigma^{-\ell}(x)$.

Some modules in $L_k(\mathfrak{sl}_2)$-wtmod



Characters of typical modules

Let k be an admissible level with $v > 1$. Then, the character of the irreducible admissible module $\mathcal{E}_{\lambda;\Delta_{r,s}}$ is given by

$$\mathrm{ch}\left[\mathcal{E}_{\lambda;\Delta_{r,s}}\right](y;z;q) = \frac{y^k z^\lambda \chi_{r,s}(q)}{\eta(q)^2} \sum_{n\in\mathbb{Z}} z^{2n}.$$

Here $\chi_{r,s}(q)$ is the character of a module of the Virasoro minimal model $M(u,v)$.
There is in fact a conformal embedding

$$L_k(\mathfrak{sl}_2) \hookrightarrow M(u,v) \otimes \Pi$$

with Π a certain extension of a Heisenberg VOA. Both $M(u,v)$ and Π admit nice semisimple Verlinde formulae.

Modularity of typical modules

Theorem (C-Ridout, 2013)

Let k be an admissible level with $v > 1$. Then, the characters of typical modules carry a (projective) representation of the modular group. Explicitly, the S-transformation is

$$\mathsf{S}\left\{\mathrm{ch}\left[\sigma^{\ell}\left(\mathcal{E}_{\lambda;\Delta_{r,s}}\right)\right]\right\} = \sum_{\ell' \in \mathbb{Z}} \sum_{r',s'}{}' \int_{-1}^{1} \mathsf{S}_{(\ell,\lambda;\Delta_{r,s})(\ell',\lambda';\Delta_{r',s'})} \mathrm{ch}\left[\sigma^{\ell'}\left(\mathcal{E}_{\lambda';\Delta_{r',s'}}\right)\right] \mathrm{d}\lambda',$$

where the S-matrix entries are given by

$$\mathsf{S}_{(\ell,\lambda;\Delta_{r,s})(\ell',\lambda';\Delta_{r',s'})} = \frac{1}{2} \frac{|\tau|}{-\mathrm{i}\tau} \mathsf{e}^{-\mathrm{i}\pi(k\ell\ell' + \ell\lambda' + \ell'\lambda)} S_{(r,s)(r',s')}$$

with the Virasoro minimal model S-matrix

$$S_{(r,s)(r',s')} = -2\sqrt{\frac{2}{uv}} (-1)^{rs'+r's} \sin \frac{v\pi rr'}{u} \sin \frac{u\pi ss'}{v}.$$

- Characters of atypical modules, if treated correctly, do not have good modular properties.
- Idea: resolve them

$$0 \longrightarrow \sigma\left(\mathcal{D}^{+}_{r,s+1}\right) \longrightarrow \sigma\left(\mathcal{E}^{+}_{r,s+1}\right) \longrightarrow \mathcal{D}^{+}_{r,s} \longrightarrow 0 \qquad (s \neq v-1),$$

$$0 \longrightarrow \sigma^2\left(\mathcal{D}^{+}_{u-r,1}\right) \longrightarrow \sigma^2\left(\mathcal{E}^{+}_{u-r,1}\right) \longrightarrow \mathcal{D}^{+}_{r,v-1} \longrightarrow 0 \qquad (s = v-1).$$

Since $\mathcal{D}^{+}_{r,v-1} \cong \sigma\left(\mathcal{L}_{u-r,0}\right)$, we also obtain

$$0 \longrightarrow \sigma\left(\mathcal{D}^{+}_{r,1}\right) \longrightarrow \sigma\left(\mathcal{E}^{+}_{r,1}\right) \longrightarrow \mathcal{L}_{r,0} \longrightarrow 0.$$

Resolving atypical modules

Proposition (C-Ridout, 2013)

Let k be an admissible level with $v > 1$. Then, the atypical irreducible module $\mathcal{L}_{r,0} = \mathcal{L}_{r-1}$ has the following resolution:

$$\cdots \longrightarrow \sigma^{3v-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma^{2v+2}\left(\mathcal{E}^+_{r,2}\right) \longrightarrow \sigma^{2v+1}\left(\mathcal{E}^+_{r,1}\right)$$
$$\longrightarrow \sigma^{2v-1}\left(\mathcal{E}^+_{u-r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma^{v+2}\left(\mathcal{E}^+_{u-r,2}\right) \longrightarrow \sigma^{v+1}\left(\mathcal{E}^+_{u-r,1}\right)$$
$$\longrightarrow \sigma^{v-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma^{2}\left(\mathcal{E}^+_{r,2}\right) \longrightarrow \sigma\left(\mathcal{E}^+_{r,1}\right) \longrightarrow \mathcal{L}_{r,0} \longrightarrow 0.$$

Similar for other atypical modules.

The logarithmic Verlinde conjecture

- We replaced characters by the Euler-Poincaré characters of the resolution and accordingly the S-kernel.
- This didn't converge and an obvious regularization was applied.
- A sensible Verlinde formula was conjectured.

Example

$$q^A_{\mathcal{L}_{r,0}\bullet}(\ell', \lambda', r', s') = e^{\pi i(k-2\lambda')} \sum_{s=1}^{v-1} (-1)^{s-1} \sum_{\ell=0}^{\infty} \Big(t^{s-1+2\ell(v-1)} z^{2v\ell+s} e^{-\pi i((k\ell'+\lambda')(2v\ell+s)+\ell'\lambda_{r,s})} -$$

$$t^{2(\ell+1)(v-1)-s} z^{2v(\ell+1)-s} e^{-\pi i((k\ell'+\lambda')(2v(\ell+1)-s)-\ell'\lambda_{r,s})} \Big) \frac{S^{\mathrm{Vir}}_{(r,s),(r',s')}}{S^{\mathrm{Vir}}_{(1,1),(r',s')}} =$$

$$\sum_{s=1}^{v-1} (-1)^{s-1} \frac{t^{s-1} z^s e^{-\pi i((k\ell'+\lambda')s+\ell'\lambda_{r,s})} - t^{2(v-1)-s} z^{2v-s} e^{-\pi i((k\ell'+\lambda')(2v-s)-\ell'\lambda_{r,s})}}{1-t^{2(v-1)} z^{2v} e^{-2\pi i v(k\ell'+\lambda')}} \frac{S^{\mathrm{Vir}}_{(r,s),(r',s')}}{S^{\mathrm{Vir}}_{(1,1),(r',s')}}$$

so this is a nice rational function in t, z. The limit $t, z \to 1^-$ doesn't depend on the order and is a rational function in trigonometric functions

$$q^A_{\mathcal{L}_{r,0}}(\ell', \lambda', r', s') = \frac{e^{-\pi i \ell'(k+r-1)}}{2\cos(\pi\lambda') + 2(-1)^r \cos(k\pi s')} \frac{S^{\mathrm{Vir}}_{(r,s),(r',s')}}{S^{\mathrm{Vir}}_{(1,1),(r',s')}}.$$

Resolving Verlinde's formula

The set-up

- $V \hookrightarrow A$ a conformal embedding.
- $\mathcal{D}$ a semi-simple vertex tensor category of A-modules.
- $\mathcal{C}$ a vertex tensor category of V-modules.
- Various assumptions,
  1. A Verlinde formula holds for $\mathcal{D}$.
  2. V can be resolved by objects in $\mathcal{D}$ (viewed as V-modules)
  3. conditions that ensure that a sensible notion of regularized quantum dimensions exists.
  4. The map $\mathrm{Irr}(\mathcal{D}) \to \mathrm{Irr}(\mathcal{C})$, $X \mapsto \mathrm{top}(X)$ is one-to-one.
  5. There is a notion of V-twisted A-modules and one needs that every simple twisted module is in fact already a true module for the VOA A.
- Experience suggests that these assumptions hold whenever A is indecomposable as a V-module.

Verlinde algebra of quantum dimensions

- The S-kernel is a function

$$S_{\bullet,\bullet} : \mathrm{Irr}(\mathcal{D}) \times \mathrm{Irr}(\mathcal{D}) \to \mathbb{C}$$

- Quantum dimension

$$q_M : \mathrm{Irr}(\mathcal{D}) \to \mathbb{C}, \qquad N \mapsto \frac{S_{M,N}}{S_{V,N}}$$

- Using resolutions this can be extended to quantum dimensions on $\mathrm{Irr}(\mathcal{C})$

Unitarity and Verlinde's algebra of characters

- $T = \mathrm{Irr}(\mathcal{D})$ is a measure space with measure μ (T stands for typical modules)
- The S-kernel is unitary if

$$\int_T \mu(Y) \left(\int_T \mu(Z)\; S^*_{X,Y} S_{Y,Z} \mathrm{ch}\left[Z\right] \right) = \mathrm{ch}\left[X\right]$$

$$\int_T \mu(Y) \; \left(\int_T \mu(Z) S_{X,Y} S^*_{Y,Z} \mathrm{ch}\left[Z\right] \right) = \mathrm{ch}\left[X\right]$$

- Verlinde algebra of characters

$$\mathrm{ch}\left[X\right] \times_{\mathrm{Ver}} \mathrm{ch}\left[Y\right] := \int_T \mu(Z) \left(\int_T \mu(W)\; \frac{S_{X,Z} S_{Y,Z} S^*_{W,Z}}{S_{V,Z}} \mathrm{ch}\left[W\right] \right)$$

Theorem (C, 2024)

Under above set-up.

1. *The category $\mathcal{C}$ admits a* **Verlinde algebra of quantum dimensions**, *that is the linear span Q of quantum dimensions satisfies that*

$$K(\mathcal{C}) \to Q, \qquad X \mapsto q_X$$

   is a ring homomorphism.

2. *The Verlinde algebra of characters satisfies* **Verlinde's formula**, *that is*

$$\mathrm{ch}[X] \times_{\mathsf{Ver}} \mathrm{ch}[Y] = \mathrm{ch}[X \otimes_V Y]$$

   for all objects X, Y in $\mathcal{C}$.

Corollary (C, 2024)

The Verlinde conjectures for $\mathfrak{sl}_2$ at admissible level and also for the singlet algebra are true.

Example

$$[\sigma^{\ell}(\mathcal{E}_{\lambda;\Delta_{r,s}})] \times_{\mathrm{Ver}} [\sigma^{\ell'}(\mathcal{E}_{\lambda';\Delta_{r',s'}})] =$$

$$\sum_{r'',s''} N^{\mathrm{Vir}\ \ (r'',s'')}_{(r,s)(r',s')} \Big([\sigma^{\ell+\ell'+1}(\mathcal{E}_{\lambda+\lambda'-k;\Delta_{r'',s''}})]$$

$$+[\sigma^{\ell+\ell'-1}(\mathcal{E}_{\lambda+\lambda'+k;\Delta_{r'',s''}})]\Big)$$

$$+\sum_{r'',s''} \Big(N^{\mathrm{Vir}\ \ (r'',s'')}_{(r,s)(r',s'-1)} + N^{\mathrm{Vir}\ \ (r'',s'')}_{(r,s)(r',s'+1)}\Big) [\sigma^{\ell+\ell'}(\mathcal{E}_{\lambda+\lambda';\Delta_{r'',s''}})].$$

with $N^{\mathrm{Vir}\ \ (r'',s'')}_{(r,s)(r',s')}$ the well-known Virasoro fusion rules.