

The Andersen–Kashaev Volume Conjecture for FAMED Geometric Triangulations

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(joint work with Ka Ho Wong)

arXiv:2410.10776

*(also joint work with François Guéritaud and Eiichi Piguet-Nakazawa)
(and also joint work with Antonin Guilloux and Ka Ho Wong)*

Our Goal

- *Defining a **FAMED** triangulation.*
- *Proving the **Andersen–Kashaev volume conjecture** for **FAMED** geometric triangulations of knot complements.*

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- ② Topology: The **FAMED** condition on a triangulation
- ③ Geometry: the triangulations contain the **hyperbolicity**
- ④ Algebra: computing the **Teichmüller TQFT**
- ④ Analysis: the **hyperbolic volume** appears **asymptotically**

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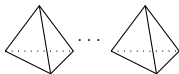
Illustrating example: the **figure-eight knot complement** $S^3 \setminus 4_1$

TOPOLOGY



Diagram
of a knot K

Triangulation
of $S^3 \setminus K$



TOPOLOGY

HYPERBOLIC GEOMETRY

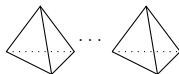


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**Hyperbolic
volume**

$\text{vol}(S^3 \setminus K)$



TOPOLOGY

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$$J_K(N, q)$$

Colored Jones
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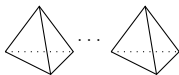
**Teichmüller
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$$Z_b(S^3 \setminus K)$$



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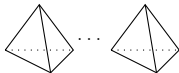


TOPOLOGY

QUANTUM
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TH 1

VOLUME
CONJECTURES

$$q = e^{2i\pi/N}, N \rightarrow \infty$$

→ Sums

Integrals

→ Integrals

Saddle point
method

$$\sim e^{N \frac{\text{vol}(S^3 \setminus K)}{2\pi}}$$

Hyperbolic
volume

$$\sim e^{\frac{1}{b^2} \frac{-\text{vol}(S^3 \setminus K)}{2\pi}}$$

$$b \rightarrow 0^+$$

TH 2

Andersen–Kashaev '11: **Teichmüller TQFT** of a **triangulated** 3-manifold M , an "**infinite-dimensional TQFT**".

Its **partition function** $\{Z_{\mathbf{b}}(M) \in \mathbb{C}\}_{\mathbf{b} > 0}$ yields an **invariant**.

Volume Conjecture (Andersen–Kashaev '11)

*The **hyperbolic volume** $\text{Vol}(S^3 \setminus K)$ appears as an **exponential decay rate** in $Z_{\mathbf{b}}(S^3 \setminus K)$ for the limit $\mathbf{b} \rightarrow 0^+$.*

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Andersen–Kashaev '11: Proof for 4_1 and 5_2 .

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Theorem (B.A.–Guéritaud–Piguet–Nakazawa '20)

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Piguet–Nakazawa '21: Proof for integral DF of the Whitehead link.

Uemura '23: Proof for 7_3 .

B.A.–Guéritaud '25+: Proof for $\Sigma_{1,1}$ -bundles over S^1 .

Theorem (B.A.–Wong '24)

*Stronger Conjectures hold for **FAMED geometric triangulations**.*

What is a **FAMED** triangulation ?

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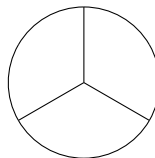
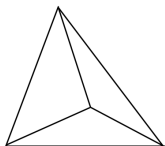
It stands for

Face **A**dacency **M**atrices with **E**dge **D**uality

It means there is a **matrix relation** between matrices associated to the **faces** and the ones associated to **edges**, hinting at a natural **duality** property.

We will define **FAMED** with the **figure-eight knot** in the following two parts.

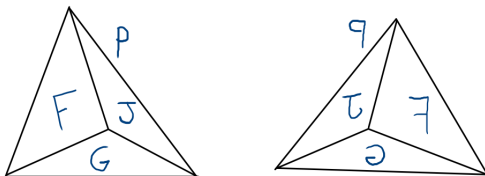
A **tetrahedron** = compact, truncated or **ideal** (without vertices).



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A **triangulation** $X = (T_1, \dots, T_N, \sim)$ of a 3-manifold $M = N$ **tetrahedra** and a **gluing relation** $\sim$ of faces pairwise.

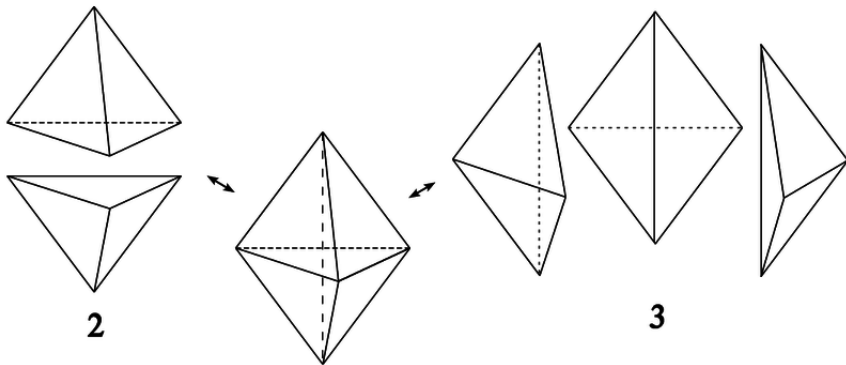


Example : $(T_1, T_2, \sim)$ triangulates either S^3 (compact T_i), $(S^3 \setminus 4 \text{ points})$ (**ideal** T_i) or $(S^3 \setminus 4 \text{ balls})$ (truncated T_i).

(2, 3)-**Pachner moves** are moves between **ideal triangulations**.

Matveev-Piergallini: X and X' triangulate the **same** M if and only if they are related by a **finite sequence** of Pachner moves.

⇒ Useful for constructing **topological invariants** for M .



source of the picture: Wikipedia

Our **tetrahedra** have **ordered vertices** ($\Rightarrow$ **oriented edges** too).
 $\leadsto$ two possible **signs** $\epsilon(T) \in \{\pm\}$. $\rightarrow$ diagonal **sign matrix** $\mathcal{E}$

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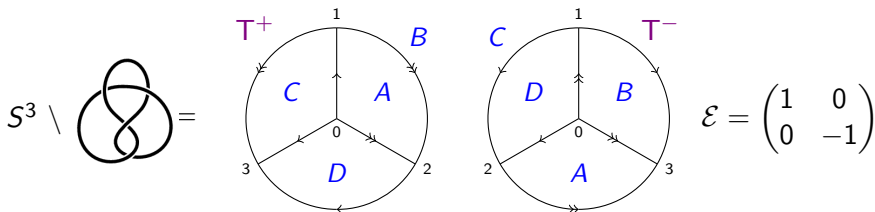
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A **triangulation** $X = (T_1, \dots, T_N, \sim)$ of a 3-manifold M is the datum of N **tetrahedra** and a **gluing relation** $\sim$ pairing their faces while **respecting the vertex order**.

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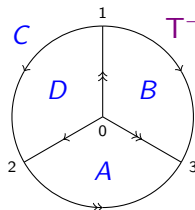
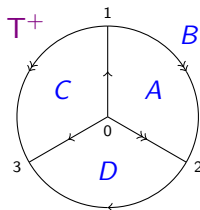
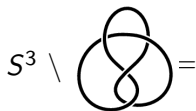
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$$X^3 = \{T^+, T^-\}, \quad X^2 = \{A, B, C, D\}, \quad X^1 = \{\rightarrow, \rightrightarrows\}, \quad X^0 = \{.\}$$

face maps $x_0, \dots, x_3: X^3 \rightarrow X^2$, for example $x_0(T^+) = B$.



$$\mathcal{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Face matrices:

$$\mathcal{X}_0 = \begin{matrix} A & B & C & D \\ T_+ & (0 & 1 & 0 & 0) \\ T_- & (0 & 0 & 1 & 0) \end{matrix}$$

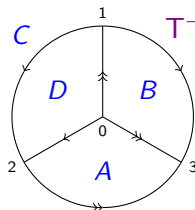
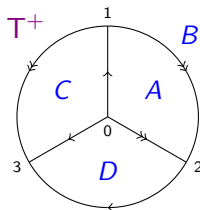
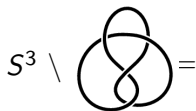
$$\mathcal{X}_1 = \begin{matrix} A & B & C & D \\ T_+ & (0 & 0 & 0 & 1) \\ T_- & (1 & 0 & 0 & 0) \end{matrix}$$

$$\mathcal{X}_2 = \begin{matrix} A & B & C & D \\ T_+ & (0 & 0 & 1 & 0) \\ T_- & (0 & 1 & 0 & 0) \end{matrix}$$

$$\mathcal{X}_3 = \begin{matrix} A & B & C & D \\ T_+ & (1 & 0 & 0 & 0) \\ T_- & (0 & 0 & 0 & 1) \end{matrix}$$

$$\mathcal{A} = \begin{pmatrix} \mathcal{X}_0 - \mathcal{X}_1 + \mathcal{X}_2 \\ \mathcal{X}_2 - \mathcal{X}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$



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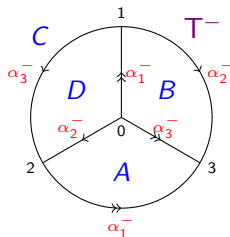
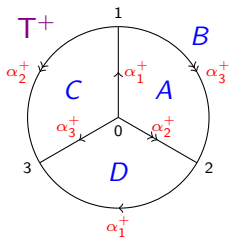
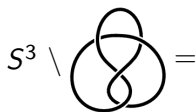
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FAMED (first half): If $\mathcal{A}$ is invertible, then we compute

$$\mathcal{X}_0 \mathcal{A}^{-1} \mathcal{B} \mathcal{E} + (\mathcal{X}_0 \mathcal{A}^{-1} \mathcal{B} \mathcal{E})^T + \frac{\mathcal{E} + \text{Id}}{2} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$\mathcal{A}_X$ is the space of **angle structures** on $X = (T_1, \dots, T_N, \sim)$, i.e. of $3N$ -tuples $\alpha \in (0, \pi)^{3N}$ of **dihedral angles** on edges, such that the angle sum is π at each vertex and 2π around each edge.

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$$\mathcal{A}_X = \left\{ \alpha = \begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ \alpha_3^+ \\ \alpha_1^- \\ \alpha_2^- \\ \alpha_3^- \end{pmatrix} \in (0, \pi)^6 \left| \begin{array}{l} \alpha_1^+ + \alpha_2^+ + \alpha_3^+ = \pi \\ \alpha_1^- + \alpha_2^- + \alpha_3^- = \pi \\ (\rightarrow) 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi \\ (\Rightarrow) 2\alpha_2^+ + \alpha_3^+ + 2\alpha_1^- + \alpha_3^- = 2\pi \end{array} \right. \right\} \ni \begin{pmatrix} \frac{\pi}{3} \\ \vdots \\ \frac{\pi}{3} \end{pmatrix}$$

α fixed $\leadsto$ angle maps $\alpha_1, \alpha_2, \alpha_3: X^3 \rightarrow \mathbb{R}$, for example $\alpha_2(T^+) = \alpha_2^+$.

The 3-dimensional **hyperbolic space** is $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2},$$

a metric which has constant curvature -1 .

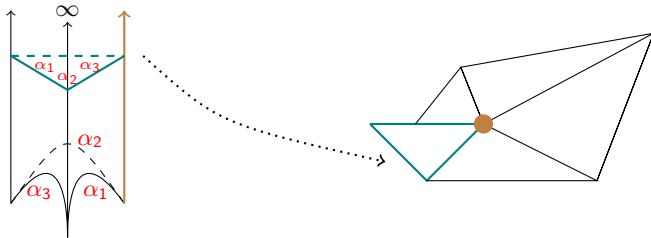
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A knot is **hyperbolic** if its complement M can be endowed with a complete hyperbolic metric of finite **volume** $\text{Vol}(M)$.

$\leadsto$ a specific $\alpha \in \mathcal{A}_X$ on $X = (T_1, \dots, T_N, \sim)$ triangulation of M .



$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

$$T \hookrightarrow \mathbb{H}^3$$

$$\sum_{\text{edge}} \alpha_j = 2\pi \quad (+ \text{others})$$

gluing gives a **manifold**

Up to isometry, we can assume that an **ideal tetrahedron** in $\mathbb{H}^3$ has vertices $\infty, 0, 1$ and $z \in \mathbb{C}$.

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Equivalent parametrisations :

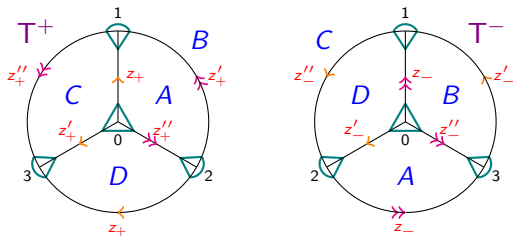
$$(\alpha_1, \alpha_2, \alpha_3) \in (0, \pi)^3 \quad \longleftrightarrow \quad z = e^{i\alpha_1} \frac{\sin(\alpha_2)}{\sin(\alpha_3)} \in \mathbb{R} + i\mathbb{R}_{>0}$$

such that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$

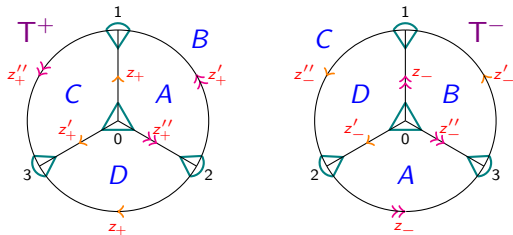
Define $z' := \frac{1}{1-z} = e^{i\alpha_2} \frac{\sin(\alpha_3)}{\sin(\alpha_1)}$ and $z'' := \frac{z-1}{z} = e^{i\alpha_3} \frac{\sin(\alpha_1)}{\sin(\alpha_2)}$,
two complex shapes associated to **isometric** tetrahedra.

→ we parametrise **edges** with z, z', z'' instead of $\alpha_1, \alpha_2, \alpha_3$.

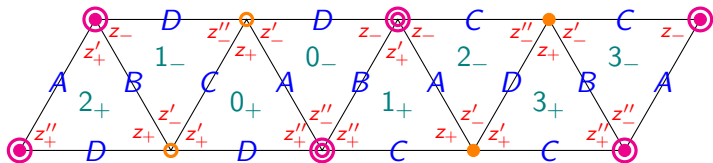
We parametrise edges with z, z', z'' and we **truncate vertices**.



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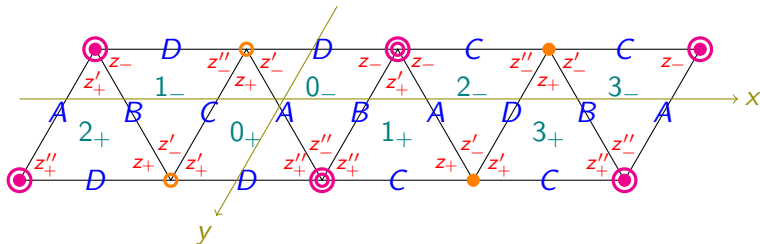


Green triangles give a **triangulation of the boundary torus** $\partial\nu K$.



($\rightarrow$) starts at $\bullet$ and ends at $\bullet$: $2\text{Log}(z_+) + \text{Log}(z'_+) + 2\text{Log}(z'_-) + \text{Log}(z''_-) = 2i\pi$

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Holonomy of boundary curve $\gamma := \sum \pm \text{Log}(z_i)$ for z_i circled by γ .

Holonomies vanish for the **complete hyperbolic structure**:

Meridian y : $-\text{Log}(z'_-) + \text{Log}(z''_+) = 0$

Preferred longitude $x + 2y$: $2i\pi - 4\text{Log}(z'_-) - 2\text{Log}(z''_-) = 0$

Neumann-Zagier matrices of the edge/completeness equations:

$$G = \begin{matrix} z_+ & z_- \\ (\rightarrow) & \begin{pmatrix} 2 & 0 \\ x+2y & 0 \end{pmatrix} \end{matrix}, \quad G' = \begin{matrix} z'_+ & z'_- \\ (\rightarrow) & \begin{pmatrix} 1 & 2 \\ x+2y & -4 \end{pmatrix} \end{matrix}, \quad G'' = \begin{matrix} z''_+ & z''_- \\ (\rightarrow) & \begin{pmatrix} 0 & 1 \\ x+2y & -2 \end{pmatrix} \end{matrix}$$

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The choice of order (z, z', z'') for each T is fixed by **X ordered**.

FAMED (second half): **If $G'' - G'$ is invertible**, then we compute

$$(G'' - G')^{-1} (G - G') = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

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We say that an ordered triangulation X is **FAMED** if

$$\mathcal{X}_0 \mathcal{A}^{-1} \mathcal{B} \mathcal{E} + (\mathcal{X}_0 \mathcal{A}^{-1} \mathcal{B} \mathcal{E})^T + \frac{\mathcal{E} + \text{Id}}{2} = (G'' - G')^{-1} (G - G')$$

Question : Does every $M = S^3 \setminus K$ admit a FAMED triangulation?

A knot is **hyperbolic** if its complement M can be endowed with a complete hyperbolic metric of finite **volume** $\text{Vol}(M)$.

The triangulation X of $S^3 \setminus K$ is called **geometric** if it admits an angle structure $\alpha^0 \in \mathcal{A}_X$ corresponding to the **complete** hyperbolic structure on $S^3 \setminus K$.

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X **geometric** $\Leftrightarrow \exists$ solution to the **nonlinear** gluing equations of X (difficult!)

Casson-Rivin, Futer-Guéritaudo: approach via $\mathcal{A}_X$, the solutions to the **linear** part: maximising the **volume functional**.

Dilogarithm function: $\text{Li}_2(z) = - \int_0^z \log(1-u) \frac{du}{u}$ for $z \in \mathbb{C} \setminus [1, \infty)$.

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Volume functional $\text{Vol}: \mathcal{A}_X \rightarrow \mathbb{R}_{\geq 0}$ (**strictly concave**) is:

$$\text{Vol}(\alpha) := \sum_{T \in X^3} \Im \text{Li}_2(z(T)) + \arg(1 - z(T)) \log |z(T)|,$$

where $z(T) = \left(\frac{\sin \alpha_3(T)}{\sin \alpha_2(T)} \right)^{\epsilon(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0}$ encodes the **angles** of T .

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Volume functional $\text{Vol}: \mathcal{A}_X \rightarrow \mathbb{R}_{\geq 0}$ (**strictly concave**) is:

$$\text{Vol}(\alpha) := \sum_{T \in X^3} \Im \text{Li}_2(z(T)) + \arg(1 - z(T)) \log |z(T)|,$$

where $z(T) = \left(\frac{\sin \alpha_3(T)}{\sin \alpha_2(T)} \right)^{\epsilon(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0}$ encodes the **angles** of T .

How to prove geometricity (Casson–Rivin):

- Check that the open polyhedron $\mathcal{A}_X$ is **non-empty**.
- General fact: the **complete** structure α^0 exists $\Leftrightarrow \max_{\overline{\mathcal{A}_X}} \text{Vol}$ is reached in $\mathcal{A}_X$.
- Prove that $\max_{\overline{\mathcal{A}_X}} \text{Vol}$ cannot be on $\partial \mathcal{A}_X$ (case-by-case).

$S(\mathbb{R}^n)$ = rapidly decreasing functions $f: \mathbb{R}^n \rightarrow \mathbb{C}$.

$S'(\mathbb{R}^n)$ = dual of $S(\mathbb{R}^n)$, **tempered distributions**.

Example: $X^2 = \{A, B\}$, **Dirac delta function**

$\delta(A) \in S'(\mathbb{R}^{X^2}) \cong S'(\mathbb{R}^2)$ acts by: $\forall f \in S(\mathbb{R}^2)$,

$$\delta(A) \cdot f = \iint_{(A,B) \in \mathbb{R}^2} dA dB \delta(A) f(A, B) = \int_{B \in \mathbb{R}} dB f(0, B) \in \mathbb{C}.$$

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⚠ Product of Dirac deltas is sometimes but not always **defined**.

$\delta(A)\delta(A)$ is not defined (because of **linear dependance**).

$\delta(A+B)\delta(A-B) = \frac{1}{2}\delta(A)\delta(B) = (f \mapsto \frac{1}{2}f(0,0))$ is well-defined.

Partition function for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X, \alpha) = \int_{\bar{x} \in \mathbb{R}^{x^2}} d\bar{x} \prod_{T_1, \dots, T_N} p_{\mathbf{b}}(T)(\alpha)(\bar{x}) \in \mathbb{C}.$$

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Tetrahedral operator: $p_{\mathbf{b}}(T)(\alpha)(\bar{x}) \in S'(\mathbb{R}^{x_2})$ is equal to

$$\frac{\delta(x_0(T) - x_1(T) + x_2(T)) e^{(2\pi i \epsilon(T)x_0(T) + (\mathbf{b} + \mathbf{b}^{-1})\alpha_3(T))(x_3(T) - x_2(T))}}{\Phi_{\mathbf{b}} \left((x_3(T) - x_2(T)) - \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2\pi} \epsilon(T)(\alpha_2(T) + \alpha_3(T)) \right)^{\epsilon(T)}}.$$

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Faddeev's quantum dilogarithm:

$$\Phi_{\mathbf{b}}(x) := \exp\left(\int_{z \in \mathbb{R} + i0^+} \frac{e^{-2izx} dz}{4 \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1})z}\right).$$

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Proposition (Andersen-Kashaev '11)

$|Z_{\mathbf{b}}(X, \alpha)|$ is **invariant** under angled Pachner moves on (X, α) .

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Volume Conjecture (Andersen-Kashaev '11)

Let X be a triangulation of a **hyperbolic knot complement** M .

- (1) $\exists \lambda_X$ linear combination of dihedral angles, $\exists$ smooth function $J_X: \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{C}$ such that $\forall$ angle structures α , $\forall \mathbf{b} > 0$,

$$|Z_{\mathbf{b}}(X, \alpha)| = \left| \int_{x \in \mathbb{R}} J_X(\mathbf{b}, x) e^{-(\mathbf{b} + \mathbf{b}^{-1})x \lambda_X(\alpha)} dx \right|.$$

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- (2) The **hyperbolic volume** $\text{Vol}(M)$ is obtained as the following semi-classical limit:

$$\lim_{\mathbf{b} \rightarrow 0^+} 2\pi \mathbf{b}^2 \log |J_X(\mathbf{b}, \mathbf{0})| = -\text{Vol}(M).$$

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Theorem (TH1, B.A.-W. '24)

(1) is proven for **all FAMED triangulations**.

Theorem (TH2, B.A.-W. '24)

(2) is proven for **all FAMED geometric triangulations**.

Proof of TH1, easiest example: For $K = 4_1$, we find $Z_{\mathbf{b}}(X, \alpha) =$

$$\iiint \frac{dA dB dC dD}{e^{(2\pi i B + (\mathbf{b} + \mathbf{b}^{-1})\alpha_3^+)(C-A)} e^{(-2\pi i C + (\mathbf{b} + \mathbf{b}^{-1})\alpha_3^-)(B-D)}} \frac{\delta(B-D+C) \delta(C-A+B) \Phi_{\mathbf{b}}\left(D-B + \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2\pi}(\alpha_2^- + \alpha_3^-)\right)}{\Phi_{\mathbf{b}}\left(A-C - \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2\pi}(\alpha_2^+ + \alpha_3^+)\right)}.$$

Then we change the variables: $2x = B + C + \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2\pi}(\alpha_1^+ - \alpha_1^-)$,
 $2y = B - C + \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2\pi}(\alpha_1^+ + \alpha_1^- - 2\pi)$ and $A = D = B + C$.

Thus, by taking the module, $|Z_{\mathbf{b}}(X, \alpha)| =$

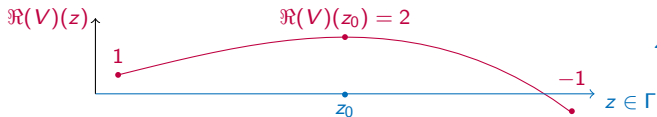
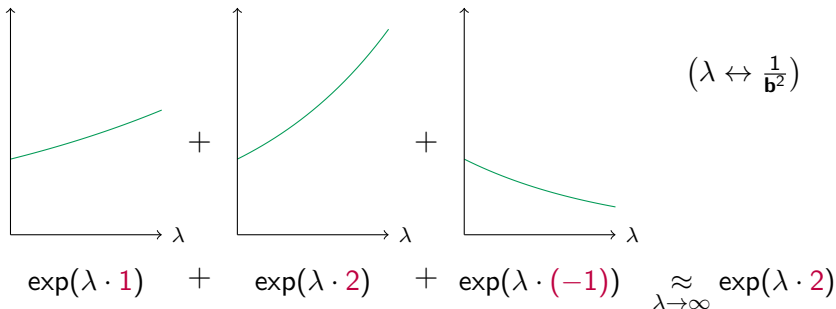
$$\left| \iint \frac{dx dy}{e^{-8\pi i xy}} \frac{\Phi_{\mathbf{b}}(x+y)}{\Phi_{\mathbf{b}}(x-y)} e^{-(\mathbf{b} + \mathbf{b}^{-1})((2\alpha_2^+ + \alpha_3^+)(x+y) + (2\alpha_2^- + \alpha_3^-)(x-y))} \right|$$

Finally we obtain (1) via $(\rightarrow) 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi$, with

$$J_X(\mathbf{b}, x) = \int_{y \in \Gamma} dy e^{8\pi i xy} \frac{\Phi_{\mathbf{b}}(x+y)}{\Phi_{\mathbf{b}}(x-y)} \quad \text{and} \quad \lambda_X(\alpha) = 4\alpha_2^+ + 2\alpha_3^+.$$

The **saddle point method** gives (under technical conditions)
asymptotics of complex **integrals with parameters** of the form:

$$\left| \int_{\Gamma} \exp \left(\frac{1}{b^2} V(z) \right) dz \right| \underset{b \rightarrow 0^+}{\approx} \exp \left(\frac{1}{b^2} \Re(V)(z_0) \right).$$



Theorem (TH2, B.A.-W. '24)

If X is a **FAMED** geometric triangulation of $S^3 \setminus K$, then:

$$\lim_{\mathbf{b} \rightarrow 0^+} 2\pi \mathbf{b}^2 \log |J_X(\mathbf{b}, 0)| = -\text{Vol}(S^3 \setminus K).$$

Sketch of proof: (a) Semi-classical approximation:

$$|J_X(\mathbf{b}, 0)| \underset{\mathbf{b} \rightarrow 0^+}{\approx} \left| \int_{\Gamma} \exp \left(\frac{1}{\mathbf{b}^2} V(z) \right) dz \right|.$$

comes from $\log \Phi_{\mathbf{b}} \underset{\mathbf{b} \rightarrow 0^+}{\approx} \text{Li}_2 + \text{technical error bounds}$

(b) Saddle point method:

$$\left| \int_{\Gamma} \exp \left(\frac{1}{\mathbf{b}^2} V(z) \right) dz \right| \underset{\mathbf{b} \rightarrow 0^+}{\approx} \exp \left(\frac{1}{\mathbf{b}^2} \Re(V)(z_0) \right).$$

we check that z_0 exists thanks to **geometricity** of X .

(c) Finally, $\Re(V)(z_0) = -\frac{1}{2\pi} \text{Vol}(S^3 \setminus K)$, from $\text{Li}_2 \leftrightarrow \text{Vol}$.

Ongoing projects:

(w/ Guilloux-Wong) **Computer**: $\exists X$ FAMED for **42000 knots**.

(w/ Guéritaud) Proof for **fibered** M^3 with fiber a punctured torus

(w/ Baseilhac) Vol Conj for **BB invariants** for **twist knots**

Future possible directions:

New formulations of Teichmüller TQFT (links, unordered X)

Apply **FAMED/geometric** triangulations to **other Vol Conj**

Prove the full **invariance** of the Teichmüller TQFT.

Hope: There always exists a FAMED geometric triangulation
 $\Rightarrow$ the volume conjecture is true !