BPS Black Holes in $AdS_3 \times S^3 \times S^3 \times S^1$ and Beyond

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Work with J.Turiaci, X. Shi

Plan for Today:

- How can near-extremal black holes be understood quantum mechanically?

- The spectrum of BPS black holes and the small N=4 supersymmetric Schwarzian theory.

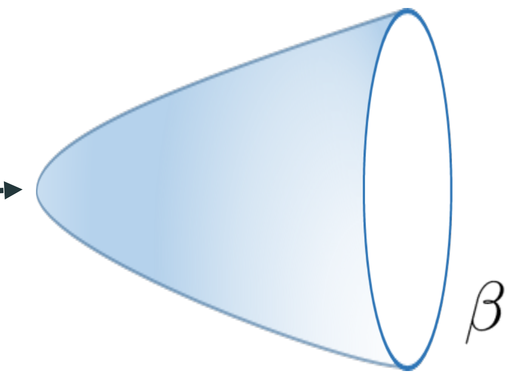
- Classification of supersymmetric black holes.

- The exceptional example: Black holes in AdS3 x S3 x S3 x S1 and new predictions for the BPS spectrum.

Gibbons-Hawking and the Path Integral

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The horizon is the tip of the “cigar”. There is no usual notion of time, no interior. Instead of the temperature being free, it is fixed by requiring spacetime is smooth.



Gibbons Hawking Proposal, Einstein-Hilbert action is a free energy:

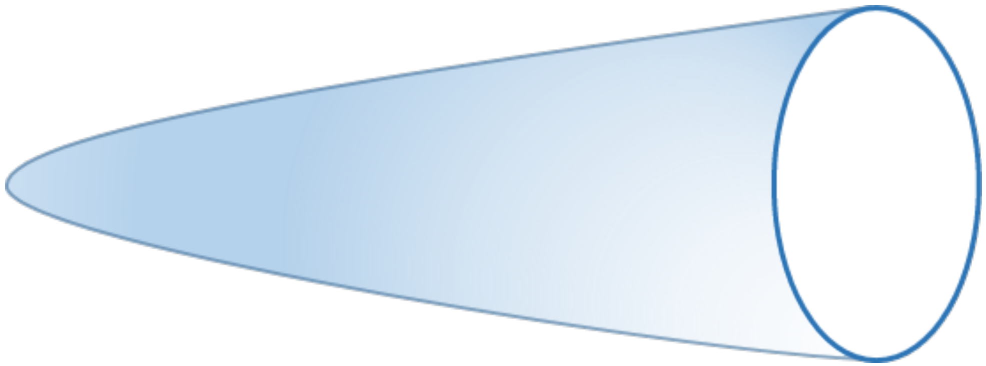
$$Z(\beta) \approx e^{-I_{\text{classical}}} + \text{subleading geometries}$$

$$\begin{aligned} I_{\text{classical}} &= \text{Classical Einstein-Hilbert action} \\ &\sim \int d^4x \text{ (curvature)} + \int_\partial d^3x \text{ (curvature of boundary)} \\ &= \beta E - S \end{aligned}$$

Adding Electric Charge

Adding electric charge to the black hole backreacts on Einstein's equation, causing the horizon to move further away:

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{Q^2G}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2G}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



To avoid a naked singularity, there is a minimal possible mass:

$$M \geq M_{ext}(Q) = |Q|$$

$$T = \frac{1}{2\pi}\frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}$$

A black hole which saturates this bound is called extremal.

They have vanishing Hawking temperature and very special properties

$$T = 0\,, \quad S_0 = \pi Q^2$$

Breakdown of Hawking’s Formula

Breakdown of Thermodynamics Near Extremality

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LIMITATIONS ON THE STATISTICAL DESCRIPTION OF BLACK HOLES

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We argue that the description of a black hole as a statistical (thermal) object must break down as the extreme (zero-temperature) limit is approached, due to uncontrollable thermodynamic fluctuations. For the recently discovered charged dilaton black holes, the analysis is significantly different, but again indicates that a statistical description of the extreme hole is inappropriate. These holes invite a more normal elementary particle interpretation than is possible for Reissner–Nordström holes.

Breakdown of Thermodynamics Near Extremality

$$E = E_0 + M_{gap}^{-1} T^2 \qquad E_{\text{Hawking}} \sim T$$

Under the assumption that the typical emitted quantum carries energy T but no charge or angular momentum (an assumption that we reconsider below), the condition for the thermal description to be self-consistent is

$$\left| T \left(\frac{\partial T}{\partial M} \right)_{Q,J} \right| \ll |T| \, . \tag{4}$$

From Eq. (1), we find

$$\left(\frac{\partial T}{\partial M} \right)_{Q,J} \simeq \frac{1}{2\pi M^2} f^{-\frac{1}{2}} \tag{5}$$

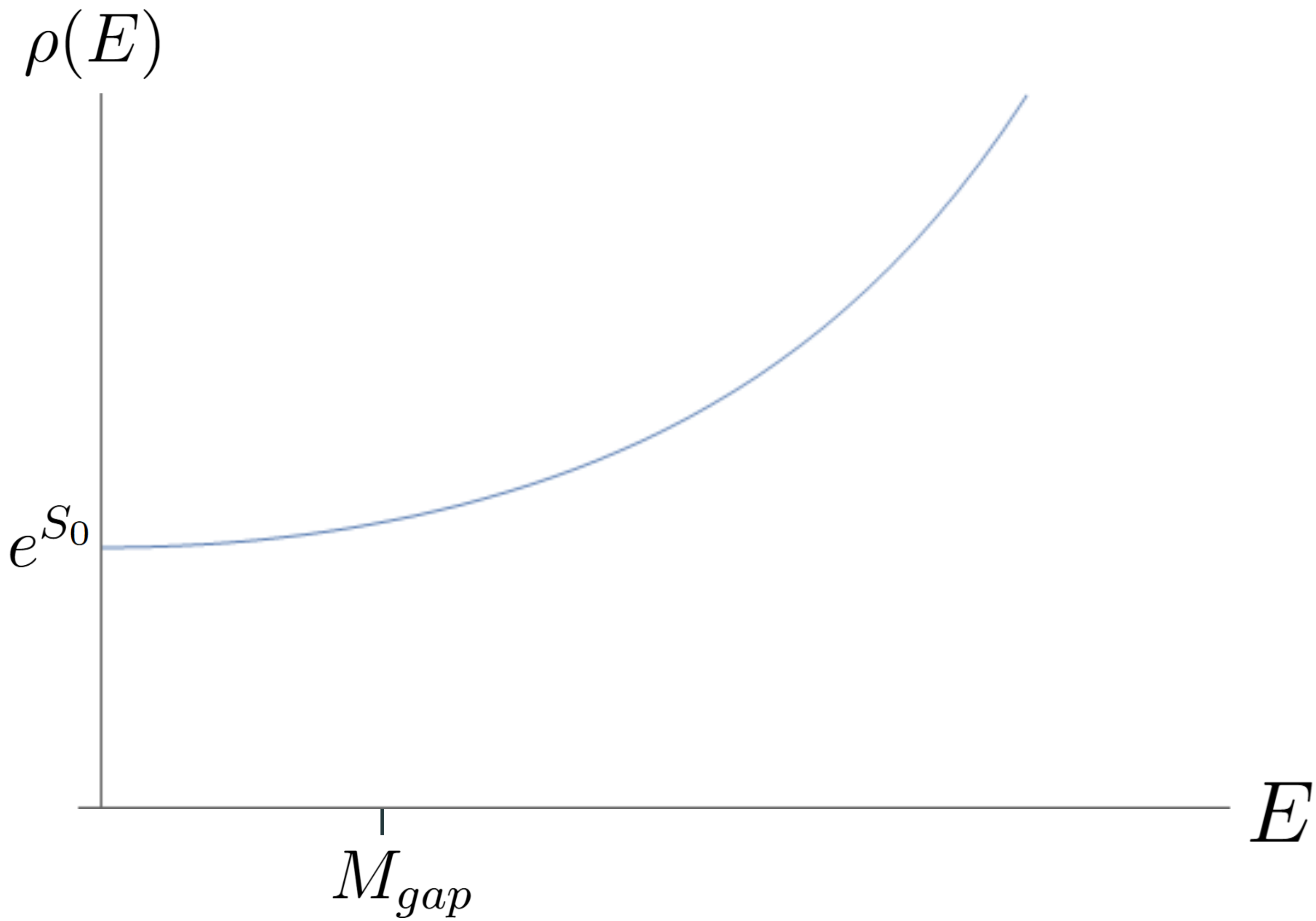
when $f \simeq 0$. Thus the thermal description of the black hole breaks down when $M^2 f^{\frac{1}{2}} \lesssim 1$.

The black hole has insufficient mass to emit a single Hawking quantum!

Paradox

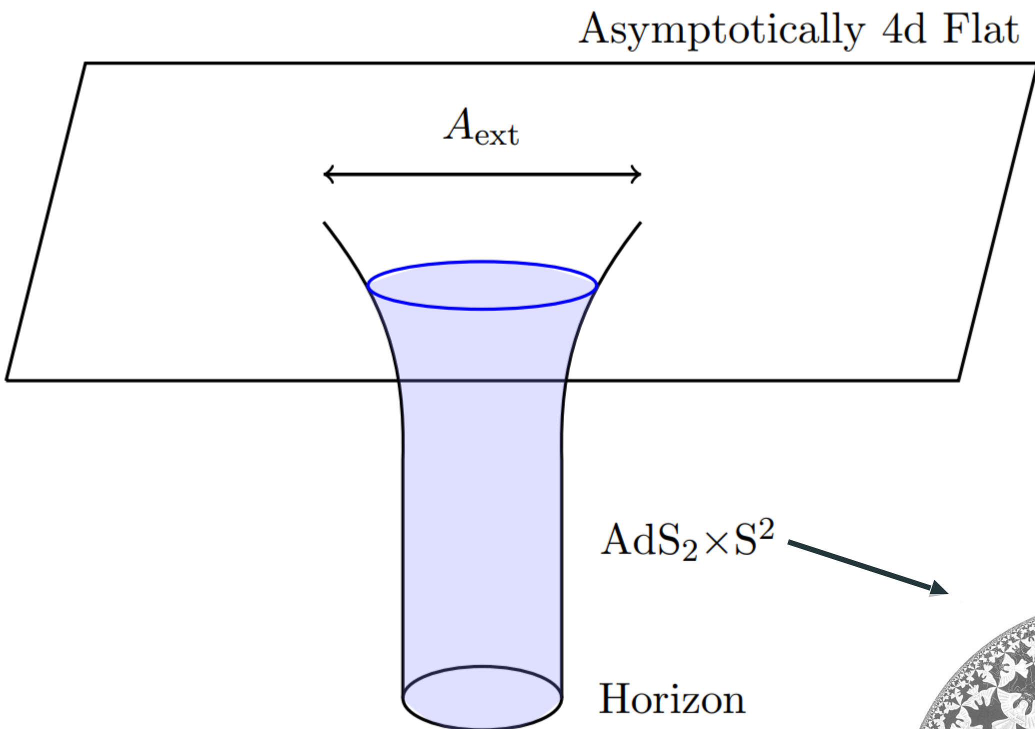
The authors propose several resolutions to this problem.

Is there a “black hole mass gap”? Does BH thermo need to be modified?

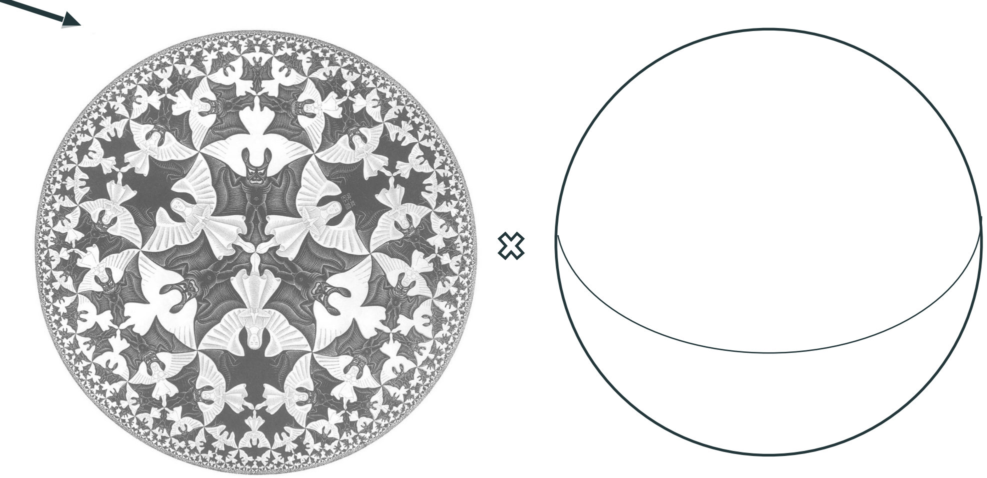


A Modern Resolution—
Quantum Gravity in the Throat

The Long Throat Region



"Universe in a box"



- Basic problem– no way to have finite energy excitations which preserve AdS2 boundary conditions. No AdS2/CFT1!

The Role of Temperature – JT Gravity

- A toy model that takes backreaction and the fluctuating boundary metric into account is known as Jackiw-Teitelboim (JT) Gravity.

$$I = \underbrace{-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}_{\text{topological term} = S_0\,\chi(\mathcal{M})} - \left[\underbrace{\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)}_{\text{gives action for boundary}}\right]$$

- Localizes on constant negative curvature metrics with a cutoff surface: $(f(\tau), z(\tau))$ The path integral reduces to an integral over $f(\tau)$ weighted by the GHY term (Maldacena, Stanford, Yang). This term evalutes to the *Schwarzian action:*

$$I = \int d\tau \mathrm{Sch}(f(\tau), \tau)\,, \quad \mathrm{Sch}(f(\tau), \tau) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

(Sachdev, Ye), (Kitaev), (Almheiri, Polchinski), (Sachdev), (Maldacena, Stanford; Yang), (Engelsoy, Mertens, H.Verlinde), (Kitaev, Suh), (Jensen), (Stanford, Witten), (Mertens, Turiaci, H.Verlinde), (Saad, Shenker, Stanford),

Asymptotics and the BF Formulation

$$e^1 = dr, \qquad e^2 = \left(\tfrac{1}{2}e^r - \mathrm{Sch}(u)e^{-r}\right) du, \qquad \omega = -\left(\tfrac{1}{2}e^r + \mathrm{Sch}(u)e^{-r}\right) du.$$

- The vanishing of the torsion tensor is not automatic, need additional lagrange multipliers for this constraint. The total action may be written as an $SL(2,R)$ BF theory.

$$A = \frac{dr}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{du}{2}\begin{pmatrix} 0 & e^r \\ -2\,\mathrm{Sch}(u)e^{-r} & 0 \end{pmatrix}.$$

$$I = -i\int_{\mathcal{M}} \mathrm{Tr}(BF) + \frac{i}{2}\int_{\partial\mathcal{M}} \mathrm{Tr}\left(BA\right).$$

- The Sch(u) appearing here is supposed to be an arbitrary function, but under bulk diffeomorphisms which maintain the asymptotic form, Sch(u) transforms as the Schwarzian derivative.

$$B + icA_u\Big|_{\mathrm{bdy}} = 0$$

Why is this 2d theory useful?

What did we miss in the classical approximation,

$$Z(\beta) \approx e^{-I_{\text{classical}}} + \text{subleading geometries}$$

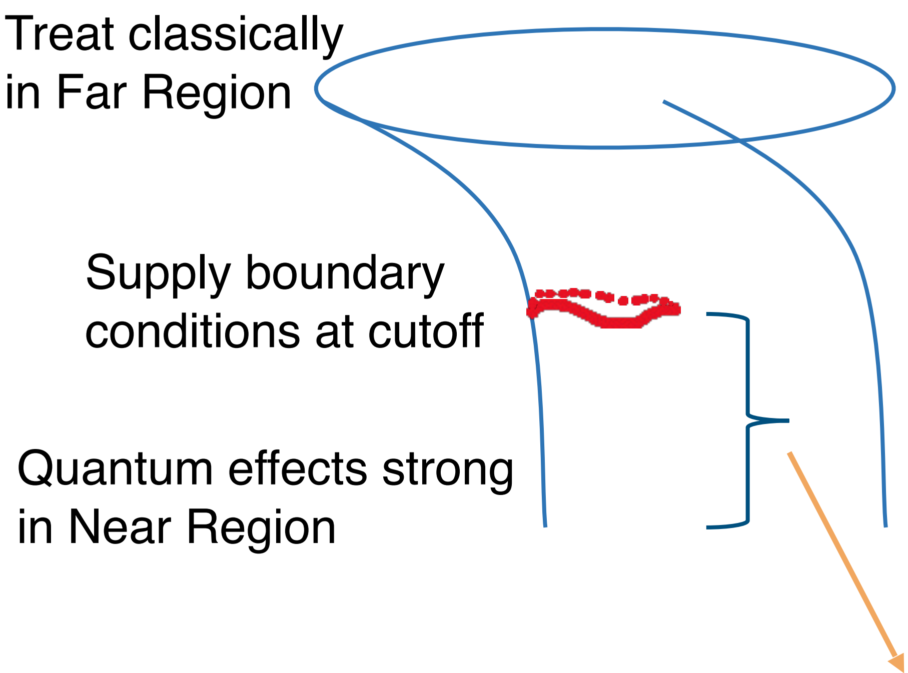
The WKB approximation says we really should have

$$Z(\beta) \approx Z_{(\text{1-loop det})} e^{-I_{\text{classical}}} + \ldots$$

This is doable in principle, and in practice. But where is the essential physics hidden?

The utility of the 2d description is that it makes it clear that Z(1-loop) is actually 1-loop exact right above extremality.

Quantum Corrections in the Throat



The key new idea is motivated by studying simple models of 2d gravity.

By expanding the Einstein action near extremality, certain modes of the metric become strongly coupled.

It is NOT appropriate to use the classical approximation for these modes.

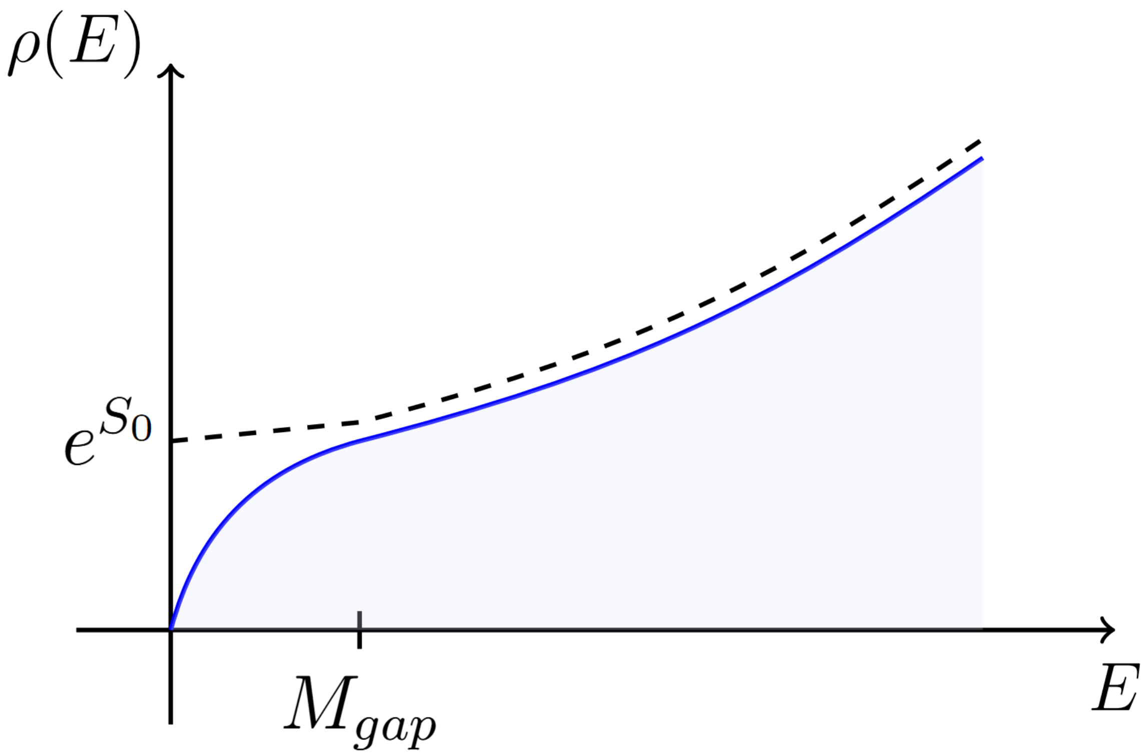
Iliesiu, Turiaci [20']; MTH, Iliesiu, Turiaci, Zhao [20']

Exact Path Integral Result via Localization

$$Z(\beta) \sim \beta^{-\frac{3}{2}} e^{S_0 + \frac{2\pi^2}{\beta M_{gap}}}$$

Exactly matches the semiclassical Hawking answer.

Each symmetry of the near horizon geometry leads to a zero mode. These must be removed for the path integral to converge. There are 3 modes which each remove a ½ power of the temperature [Stanford, Witten '17]



Black Hole Microstates in Supergravity

Supersymmetric Black Hole Microstates

Strominger and Vafa counted the number of open string bound states for a given mixed macroscopic charge using supersymmetry. Their calculation ONLY works for states that satisfy:

$$M = Q$$

The supersymmetric black holes are automatically extremal. The asymptotic growth of their formula remarkably gives:

$$S_{BH} = \frac{A}{4G}$$

But now we see the key problem: String theory validates Hawking's result in a regime where his methods should not apply!

Resolution of the B/H vs Strominger/Vafa puzzle

Consider black holes in 4d N=2 Supergravity

MTH, Iliesiu, Turiaci, Zhao [20'] + several other papers (MTH 24')

$$E^{-1}\mathcal{L} = \kappa^{-2}(\frac{1}{2}R - \bar{\Psi}_{IM}\Gamma^{MNP}D_N\Psi_P^I - \frac{1}{4}F_{MN}F^{MN} + \frac{\varepsilon^{IJ}}{2\sqrt{2}}\bar{\Psi}_I^M(F_{MN} + i\star F_{MN}\Gamma_5)\Psi_J^N + 4\text{ gravitino})\,,$$

$$ds^2 = \frac{r_0^2}{z^2}(-dt^2 + dz^2) + r_0^2 d\Omega_2^2\,, \quad F = \frac{1}{2}\frac{\kappa Q}{\sqrt{4\pi}}\frac{1}{z^2}dt\wedge dz\,.$$

- Has a $PSU(1,1\,|\,2)$ superisometry, aka the small $\mathcal{N}=4$ superconformal algebra
- This is the same supergroup preserved by extremal BTZ black holes in AdS3 x S3 x T4/K3, dual to the D1-D5 CFT

Reduction to Super-JT Gravity

$$d\hat{s}^2_{4D} = \frac{r_0}{\chi^{1/2}} g_{\mu\nu} dx^\mu dx^\nu + \chi\, h_{mn}(dy^m + \xi^m_i B^i_\mu dx^\mu)(dy^n + \xi^n_j B^j_\nu dx^\nu)\,,$$

$$\frac{\chi(x)}{G_N} = \frac{r_0^2}{G_N} + 2\Phi(x) + \mathcal{O}\left(\frac{G_N \Phi^2}{r_0^2}\right)$$

- Results in the bosonic part of $\mathcal{N} = 4$ JT supergravity:

$$S_{2D} = S_0 - \frac{1}{2}\int d^2x \sqrt{g}\,\Phi\left(R + \frac{2}{r_0^2}\right) - i\int \mathrm{tr}_{SU(2)} bH \, + \mathcal{O}\left(\frac{G_N}{r_0^2}\right)\,.$$

- May be written as a $PSU(1,1\,|\,2)$ BF theory:

e^a
ω
B^i
$\psi^\alpha,\, \bar{\psi}^\alpha$

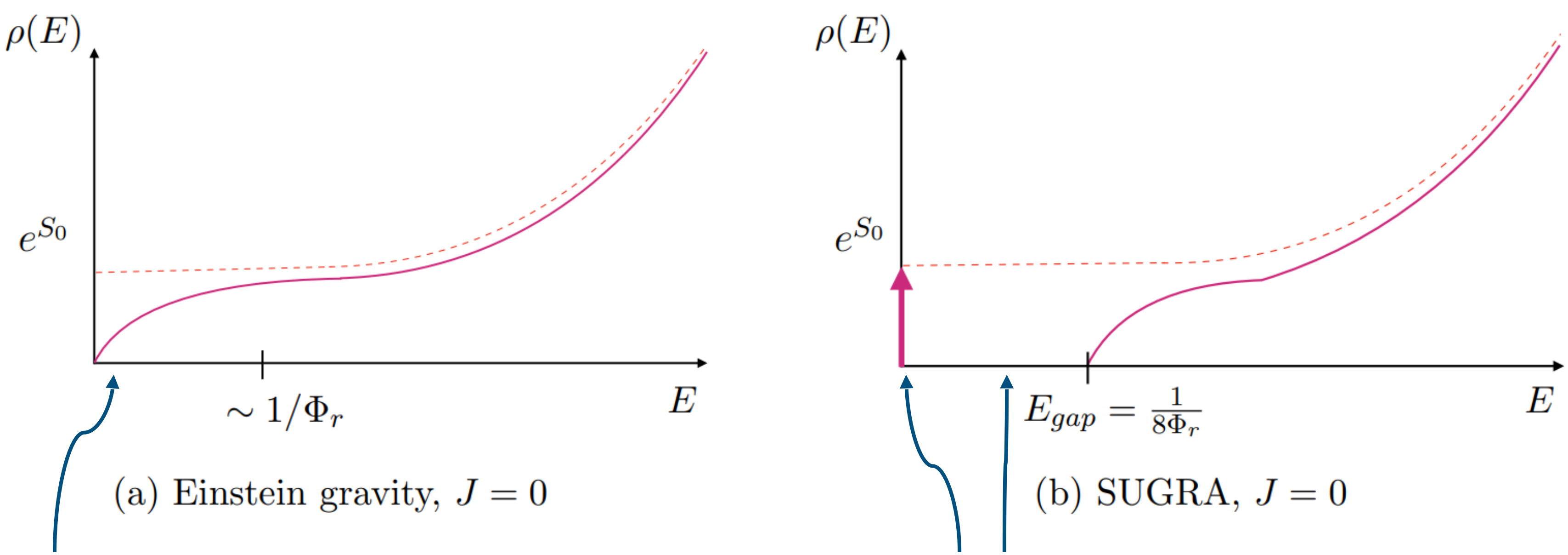
Localization of $\mathcal{N}$=4 JT Gravity

$$Z(\beta, \alpha_i) = \int \frac{\mathcal{D}f\mathcal{D}g\mathcal{D}\eta}{G} e^{-I_{\text{super-Schw}}}$$
$$= \sum_{\text{fixed points}} Z^{\text{Schw}}_{1-\text{loop}} \, Z^{R}_{1-\text{loop}} \, Z^{\text{fermions}}_{1-\text{loop}} \, e^{-I_{\text{fixed points}}}$$

$$Z(\beta, \alpha) = e^{S_0} \sum_{n\in\mathbb{Z}} \frac{\beta}{\Phi_r} \frac{2\cot(\pi\alpha)(\alpha+n)}{\pi^3(1-4(n+\alpha)^2)^2} e^{\frac{2\pi^2\Phi_r}{\beta}(1-4(n+\alpha)^2)}$$

BPS and Near BPS Black Holes

(MTH, Iliesiu, Turiaci, Zhao)



Non-perturbative completion of the spectrum in $\exp(-S_0)$ might resolve individual microstates, but might not call them extremal black holes!

Remarkable prediction of quantum (super)gravity: Discrete BPS spectrum from a gravity calculation (Strominger,Vafa). Gap is polynomial in S_0 , previously argued by (Maldacena, Strominger), (Maldacena, Susskind)

- Same general spectrum is found for AdS3xS3 black holes

Beyond $\mathcal{N} = 4$ Supersymmetry

Supergroup Extremal Black Holes

- Most/all studied black holes in string theory preserve at most 4 supercharges.
- Why is this?
- We can study the possible supersymmetric extensions of SL(2).
- For each, we can associate a supergroup BF theory / super JT gravity!

Superconformal Isometries	R-symmetry G_R	ρ	Dimensions (N_b, N_f)
$\mathrm{OSp}(n\|2)$	$\mathrm{SO}(n)$	n	$(\frac{1}{2}n^2 - \frac{1}{2}n + 3, 2n)$
$\mathrm{SU}(1,1\|n), \quad n \neq 2$	$\mathrm{SU}(n) \times \mathrm{U}(1)$	$n + \bar{n}$	$(n^2 + 3, 4n)$
$\mathrm{OSp}(4^*\|2n)$	$\mathrm{SU}(2) \times \mathrm{USp}(2n)$	$(2, 2n)$	$(2n^2 + n + 6, 8n)$
$\mathrm{PSU}(1,1\|2)$	$\mathrm{SU}(2)$	$2 + \bar{2}$	$(6,8)$
$\mathrm{D}(2,1\|\alpha)$	$\mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$	$(2, 2)$	$(9, 8)$
$\mathrm{G}(3)$	G_2	7	$(17, 14)$
$\mathrm{F}(4)$	$\mathrm{SO}(7)$	8_s	$(24, 16)$

Supergroup Extremal Black Holes

- The usual dictionary allows us to extract the Schwarzian theory starting with the superalgebra.

$$I_{BF} = -\mathrm{i} \int \mathrm{Str}\, \phi F, \quad F = \mathrm{d}A - A \wedge A, \qquad I_{BF} = \Phi_r \oint \mathrm{Str}\, A_\tau^2 .$$

$$I_{\mathrm{super-Schw}} = -\Phi_r \int \mathrm{d}\tau \left\{ \mathrm{Sch}(f, \tau) + q\, \mathrm{Tr}_\rho[(g^{-1}\partial_\tau g)^2] + \mathrm{fermions} \right\} .$$

- Using the Duistermaat-Heckman localization theorem gives the partition function:

$$Z(\beta, \alpha_i) = \sum_{\nu \in \check{T}} \left(\frac{\beta}{\Phi_r} \right)^{(N_f - N_b)/2} \prod_{r \in R_+} \frac{2\pi r \cdot (\alpha + \nu)}{\sin 2\pi r \cdot (\alpha + \nu)} \prod_{\mu \in V_{\rho +}} \frac{\cos 2\pi \mu \cdot (\alpha + \nu)}{1 - 16(\mu \cdot (\alpha + \nu))^2}$$
$$\times \exp \left[S_0 + \frac{2\pi^2 \Phi_r}{\beta} \left(1 - 8q\, \mathrm{Tr}_\rho \left[((\alpha + \nu) \cdot H)^2 \right] \right) \right] .$$

Black Holes Can Preserve at Most 4 Supercharges

- Which supergroups lead to physically sensible partition functions?

	Superconformal Isometries	R-symmetry G_R	ρ	Dimensions (N_b, N_f)
↓	$\mathrm{OSp}(n\|2)$	$\mathrm{SO}(n)$	n	$(\frac{1}{2}n^2 - \frac{1}{2}n + 3, 2n)$
↓	$\mathrm{SU}(1,1\|n), \quad n \neq 2$	$\mathrm{SU}(n) \times \mathrm{U}(1)$	$n + \bar{n}$	$(n^2 + 3, 4n)$
↓	$\mathrm{OSp}(4^*\|2n)$	$\mathrm{SU}(2) \times \mathrm{USp}(2n)$	$(2, 2n)$	$(2n^2 + n + 6, 8n)$
↑	$\mathrm{PSU}(1,1\|2)$	$\mathrm{SU}(2)$	$2 + \bar{2}$	(6,8)
↑	$\mathrm{D}(2,1\|\alpha)$	$\mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$	$(2,2)$	$(9,8)$
↓	$\mathrm{G}(3)$	G_2	7	$(17,14)$
↓	$\mathrm{F}(4)$	$\mathrm{SO}(7)$	8_s	$(24,16)$

- The first and final two entries are technically allowed in AdS3 if we allow nonlinear superconformal symmetries. Unclear how to realize these in higher-D Einstein gravity. See (Larsen, Kraus, Shah '07) for AdS3.

- $\mathrm{D}(2,1|\alpha)$ ially interesting case as it describes a plethora of new results for BTZ black holes in

$$AdS_3 \times S^3 \times S^3 \times S^1$$

Black Holes in $AdS_3 \times S^3 \times S^3 \times S^1$

$AdS_3 \times S^3 \times S^3 \times S^1$ and the D1-D5-D5 CFT

There is a remarkable background of Type IIB which realizes the large N=4 superconformal symmetry.

(Cowdall, Townsend 98; Boonstra, Peeters, Skenderis 98;

Gauntlett, Myers, Townsend 98; de Boer, Pasquinucci, Skenderis 98;

Gukov, Martinec, Moore, Strominger 04; Tong 14)

- In principle we know the brane fluxes / supergravity background required, but to this day the dual CFT remains unknown, but see (Witten 24) for a proposal.

- Basic conceptual/technical problem:

$\mathrm{D}(2,1|\alpha)$: Linear algebra, linear BPS bound

A_γ : Linear algebra, nonlinear BPS bound

$\mathcal{N} > 4$: Nonlinear algebra, nonlinear BPS bound

$AdS_3 \times S^3 \times S^3 \times S^1$ and the D1-D5-D5 CFT

The mismatch of BPS bounds due to the nonlinearity meant that the NS sector supergravity computation could never be consistent with (affine) N=4 superconformal symmetry.

This problem was resolved in:

(Eberhardt, Gaberdiel, Gopakumar, Li 17; Baggio, Sax, Sfondrini, Stefanski, Torrielli 17):

$$j_+ = j_- \ , \quad \text{(BPS states, NS sector)}$$

The Ramond sector (which contains the supersymmetric black holes) was never understood. We will use JT gravity to make a prediction for this sector.

Some of our results can be compared with (Murthy, Rangamani 25), which instead uses the worldsheet theory. We find consistent results when applicable.

The Vacuum and Central Charge

$$I = \frac{2\pi}{g_s^2}\int \sqrt{-g}\, e^{-2\Phi}\, R - \pi \int F_3 \wedge *F_3 + \ldots,$$

$$\mathrm{d}s^2 = \ell^2 \underbrace{\frac{-\mathrm{d}t^2 + \mathrm{d}\varphi^2 + \mathrm{d}z^2}{z^2}}_{\text{unit } AdS_3} + R_+^2 \underbrace{(\mathrm{d}\theta_+^2 + \sin^2\theta_+ \mathrm{d}\phi_+^2 + \cos^2\theta_+ \mathrm{d}\psi_+^2)}_{\text{unit } S_+^3}$$

$$+ R_-^2 \underbrace{(\mathrm{d}\theta_-^2 + \sin^2\theta_- \mathrm{d}\phi_-^2 + \cos^2\theta_- \mathrm{d}\psi_-^2)}_{\text{unit } S_-^3} + L^2 \underbrace{\mathrm{d}\theta^2}_{\text{unit } S^1},$$

$$F_3 = Q_1 * (\epsilon_+ \wedge \epsilon_- \wedge \mathrm{d}\theta) + Q_5^+ \epsilon_+ + Q_5^- \epsilon_-$$

$$-\frac{1}{\ell^2} + \frac{1}{R_+^2} + \frac{1}{R_-^2} = 0 \qquad R_\pm = \frac{1}{2\pi}\sqrt{g_s Q_5^\pm},$$

$$k^+ = Q_1 Q_5^+, \qquad k^- = Q_1 Q_5^-$$

$$c = \frac{6k^+k^-}{k^+ + k^-}, \qquad \alpha = \frac{k^-}{k^+} = \frac{Q_5^-}{Q_5^+} = \frac{R_-^2}{R_+^2}$$

The Large $\mathcal{N} = 4$ Superconformal Algebra

$$A_\gamma : \qquad L_n\,, \qquad G_n^{A\dot{A}}, \qquad T_n^{AB}, \qquad T_n^{\dot{A}\dot{B}}, \qquad Q_n^{A\dot{A}}, \qquad U_n$$

$$\{G_n^{A\dot{B}}, G_m^{C\dot{D}}\} = -\frac{1}{2}\epsilon^{\dot{B}\dot{D}}\epsilon^{AC}\left[2L_{n+m} + \frac{c}{3}\delta_{n+m}\left(m^2 - \frac{1}{4}\right)\right],$$

$$+(n-m)\left[\frac{k^+}{k^+ + k^-}T_{n+m}^{\dot{B}\dot{D}}\varepsilon^{AC} + \frac{k^-}{k^+ + k^-}T_{n+m}^{AC}\varepsilon^{\dot{B}\dot{D}}\right]$$

$$\{Q_n^{A\dot{B}}, Q_m^{C\dot{D}}\} = (k^+ + k^-)\varepsilon^{\dot{B}\dot{D}}\varepsilon^{AC}\delta_{n+m,0}$$

$$\{Q_n^{A\dot{B}}, G_m^{C\dot{D}}\} = T_{n+m}^{AC}\varepsilon^{\dot{B}\dot{D}} - T_{n+m}^{\dot{B}\dot{D}}\varepsilon^{AC} + \mathrm{i}\varepsilon^{AC}\varepsilon^{\dot{B}\dot{D}}U_{n+m}$$

$$[T_m^{+,I}, G_n^{A\dot{B}}] = \frac{1}{2}(\sigma^I)_C{}^A\left(G_{m+n}^{C\dot{B}} + \frac{k^+}{k^+ + k^-}mQ_{m+n}^{C\dot{B}}\right)$$

$$[U_m, G_n^{A\dot{A}}] = \mathrm{i}\frac{m}{2}Q_{m+n}^{A\dot{A}}$$

The BTZ Black Hole Solution

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + \frac{\ell^2\mathrm{d}r^2}{f} + r^2\Big[\mathrm{d}\varphi - \frac{r_- r_+}{r^2}\mathrm{d}t\Big]^2$$
$$+R_+^2\Big[\mathrm{d}\theta_+^2 + \sin^2\theta_+\Big(\mathrm{d}\phi_+ + \frac{A_L^{+\,3} + A_R^{+\,3}}{2}\Big)^2 + \cos^2\theta_+\Big(\mathrm{d}\psi_+ + \frac{A_L^{+\,3} - A_R^{+\,3}}{2}\Big)^2\Big]$$
$$+R_-^2\Big[\mathrm{d}\theta_-^2 + \sin^2\theta_-\Big(\mathrm{d}\phi_- + \frac{A_L^{-\,3} + A_R^{-\,3}}{2}\Big)^2 + \cos^2\theta_-\Big(\mathrm{d}\psi_- + \frac{A_L^{-\,3} - A_R^{-\,3}}{2}\Big)^2\Big]$$
$$+L^2\mathrm{d}\theta^2$$

$$f = (r^2 - r_+^2)(r^2 - r_-^2)/r^2$$

$$A_L^{\pm 3} = 2\Omega_L^{\pm}\frac{\mathrm{d}\varphi - \Omega\mathrm{d}t}{1-\Omega}, \qquad A_R^{\pm 3} = 2\Omega_R^{\pm}\frac{\mathrm{d}\varphi - \Omega\mathrm{d}t}{1+\Omega}$$

$$\beta = \frac{2\pi\ell r_+}{r_+^2 - r_-^2}, \quad \text{and} \quad \Omega = r_-/r_+$$

$$4\pi\mathrm{i}\alpha_\pm = \beta\Omega_L^{\pm},\ \ z_\pm = e^{4\pi\mathrm{i}\alpha_\pm}, \quad \text{and} \quad 4\pi\mathrm{i}\bar{\alpha}_\pm = \beta\Omega_R^{\pm},\ \ \bar{z}_\pm = e^{-4\pi\mathrm{i}\bar{\alpha}_\pm}$$

Extremal, BPS, and the "Nonlinear" Constraint

Extremality condition:

$$h^{ext} - \frac{c}{24} = \frac{j_-^2}{k^-} + \frac{j_+^2}{k^+}$$

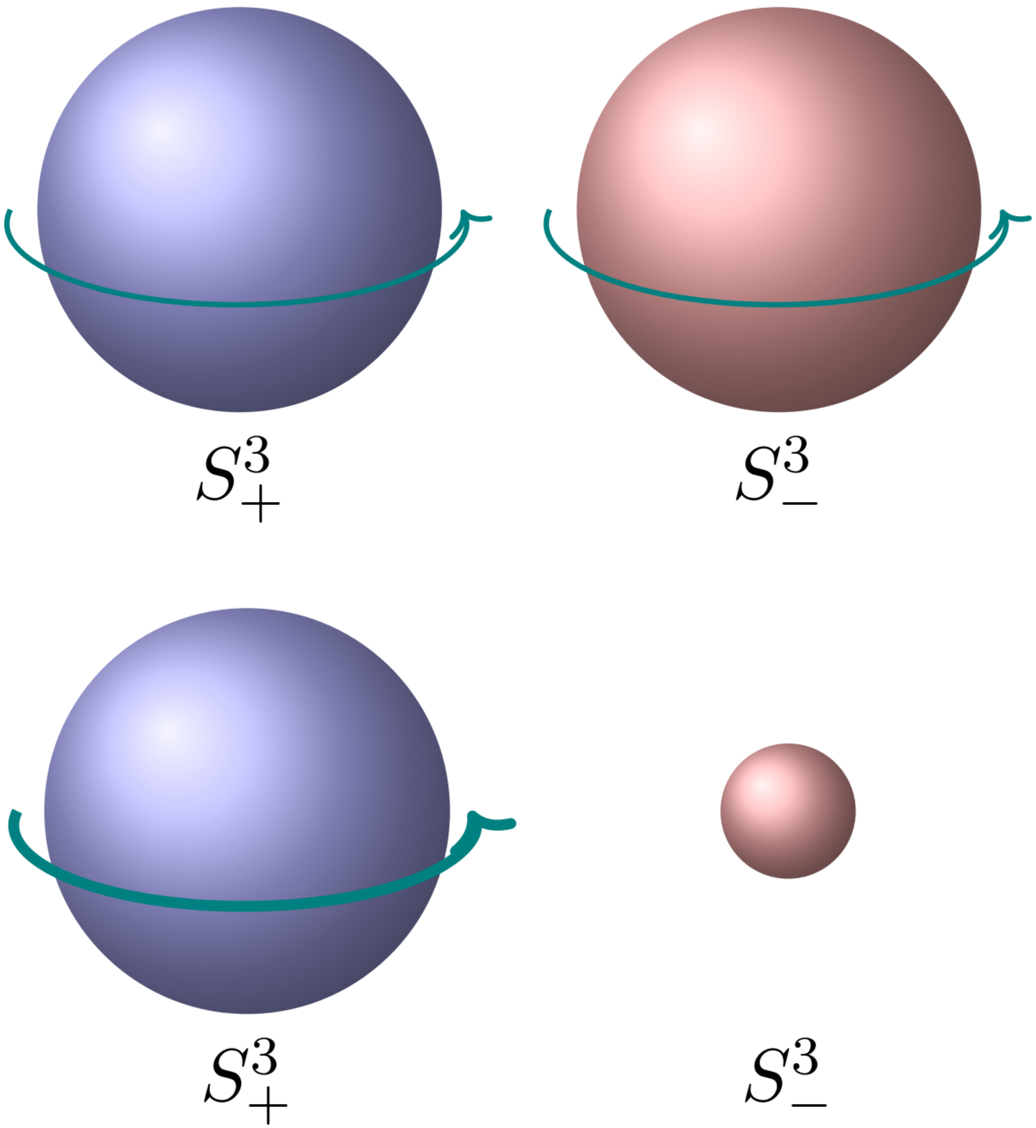
BPS condition:

$$h^{BPS} - \frac{c}{24} = \frac{(j_+ + j_-)^2}{k^+ + k^-}\,, \qquad \text{(classical)}$$

The intersection of these conditions is the analog of the "nonlinear charge relation" found in higher dimensional AdS black holes:

$$j_- = \frac{k^-}{k^+} j_+\,, \qquad \text{(classical)}$$

Distribution of BPS States



Surprises in the Ramond Sector

The Large $\mathcal{N} = 4$ Schwarzian Theory

Based on the logic of the earlier sections, we argue the Ramond sector BPS and near BPS states are captured by the appropriate Schwarzian:

$$I = S_0 - \Phi_r \int \mathrm{d}\tau \left\{ \mathrm{Sch}(f,\tau) + \frac{1}{\gamma}\,\mathrm{Tr}[(g_+^{-1}\partial_\tau g_+)^2] + \frac{1}{1-\gamma}\,\mathrm{Tr}[(g_-^{-1}\partial_\tau g_-)^2] + \frac{1}{2\gamma(1-\gamma)}(g_0^{-1}\partial_\tau g_0)^2 \right\} + (\text{fermions})\,,$$

$$\alpha = (1-\gamma)/\gamma$$

The path integral computes a partition function which depends on 4 potentials

$$Z(\beta, \alpha_+, \alpha_-, \mu) = \mathrm{Tr}\left[e^{-\beta H} e^{4\pi \mathrm{i}\alpha_+ J_+^3} e^{4\pi \mathrm{i}\alpha_- J_-^3} e^{2\pi \mathrm{i}\mu U} \right]$$

The Partition Function

We merely present the result, which follows from localization:

$$Z = \sum_{n,m\in\mathbb{Z}} \sum_{r\in\mathbb{Z}/u_0} Z_{\text{1-loop}}\, e^{S_0+\frac{2\pi^2\Phi_r}{\beta}\left(1-\frac{4(1+\alpha)}{\alpha}(\alpha_++n)^2-4(1+\alpha)(\alpha_-+m)^2-\frac{(1+\alpha)^2}{\alpha}(\mu+r)^2\right)}$$

$$Z_{\text{1-loop}} = \frac{\Phi_r}{\beta}\sqrt{\frac{2\pi(1+\alpha)^2}{\alpha}}\underbrace{\frac{(\alpha_++n)}{\sin 2\pi\alpha_+}\frac{(\alpha_-+m)}{\sin 2\pi\alpha_-}}_{\text{from SU(2) gauge fields}} \times \underbrace{4\cos\pi(\alpha_++\alpha_-)\cos\pi(\alpha_+-\alpha_-)}_{\text{from fermion partners of U(1) mode}}$$

$$\times \underbrace{\frac{\cos\pi(\alpha_++\alpha_-)\cos\pi(\alpha_+-\alpha_-)}{(1-4(\alpha_++\alpha_-+m+n)^2)(1-4(\alpha_+-\alpha_--m+n)^2)}}_{\text{from fermions partners of reparam. mode}}$$

BPS and near-BPS Density of States

A challenging computation is required to compute the Laplace transform.

The states are best organized by introducing the supercharacters of (Eguchi, Taormina)

$$Z(\beta,\alpha_+,\alpha_-,\mu) = \sum_{u\in u_0\cdot\mathbb{Z}} \sum_{j_+,j_-} \chi^{\text{long}}_{j_+j_-u}(\alpha_+,\alpha_-,\mu) \int \mathrm{d}E\, e^{-\beta E} \rho_{j_+j_-u}(E) + \sum_{u\in u_0\cdot\mathbb{Z}} \sum_{j_+,j_-} \chi^{\text{short}}_{j_+j_-u}(\alpha_+,\alpha_-,\mu) e^{-\beta E_{\text{BPS}}(j_+,j_-,u)} N_{j_+j_-u}$$

$$\rho_{j_+j_-u}(E) = \frac{e^{S_0}\alpha^{3/2}\, j_+j_-}{32\Phi_r\sqrt{2\pi}(1+\alpha)^3}\left(E - \frac{\alpha\left((j_+-j_-)^2+u^2\right)}{2\Phi_r(1+\alpha)^2}\right)^{-1}\left(E - \frac{\alpha\left((j_++j_-)^2+u^2\right)}{2\Phi_r(1+\alpha)^2}\right)^{-1} \times \sinh\left(2\pi\sqrt{2\Phi_r\left(E - \frac{\alpha j_+^2 + j_-^2 + \alpha/(1+\alpha)u^2}{2\Phi_r(1+\alpha)}\right)}\right)$$

The long multiplets form a continuum above a gap scale which depends on the charges:

$$E \geq E_0(j_+,j_-) = \frac{\alpha j_+^2 + j_-^2}{2\Phi_r(1+\alpha)} + \frac{\alpha}{2\Phi_r(1+\alpha)^2}u^2$$

Surprise in the BPS Spectrum

The nonlinear BPS bound leads to a surprising conclusion– the BPS states are not just concentrated in a single charge sector. There are instead an infinite number of superselection sectors which have different energies!

$$E_{\mathrm{BPS}}(j_-, j_+) = \frac{\alpha}{2\Phi_r(1+\alpha)^2}\left(\left(j_+ + j_- + \frac{1}{2}\right)^2 + u^2\right)$$

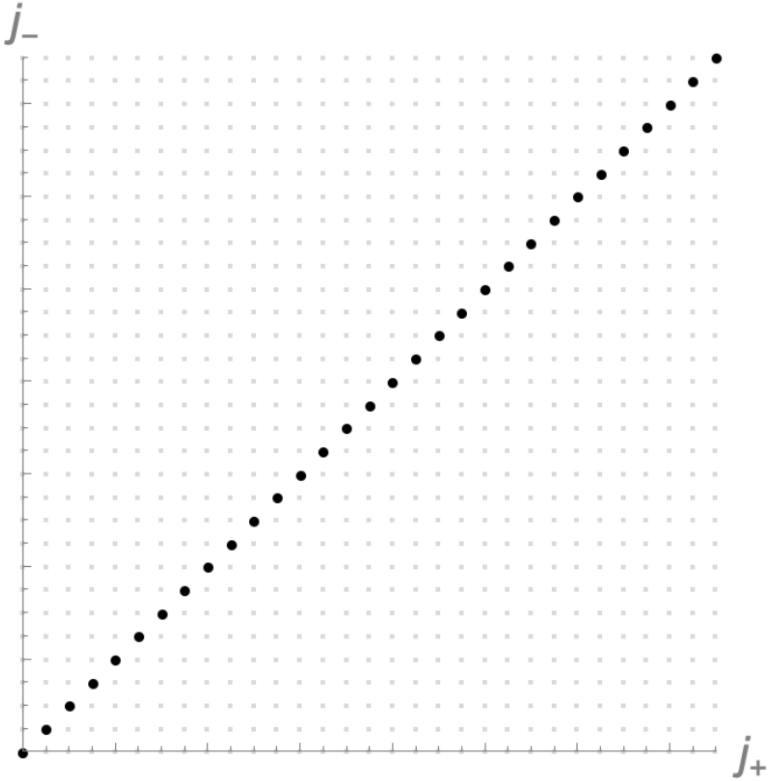
The number of BPS states for each value of j's is a dynamical question which does not follow from the algebra alone. They only exist for certain charges.

$$N_{j_+ j_- u} = \begin{cases} e^{S_0} \frac{\sqrt{\pi\alpha/2}}{32(1+\alpha)} \sin\left(\frac{2\alpha j_+ - 2j_- + \alpha}{1+\alpha}\right), & \text{if} \quad (j_+, j_-) \in R_{\mathrm{BPS}} \\ 0 & \text{if} \quad (j_+, j_-) \notin R_{\mathrm{BPS}} \end{cases}$$

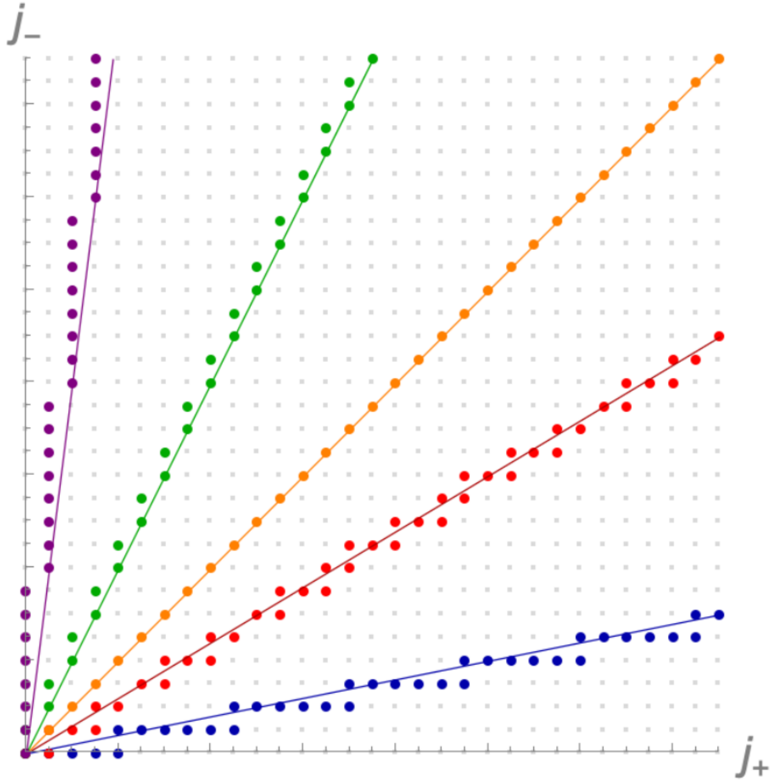
$$S_0 = 2\pi\sqrt{\frac{Q_1 Q_5^+ Q_5^-}{Q_5^+ + Q_5^-} J} \qquad R_{\mathrm{BPS}} = \left\{ (j_+, j_-) \,\middle|\, 0 < \frac{2\alpha j_+ - 2j_- + \alpha}{1+\alpha} < 1 \right\}$$

This is a quantum corrected version of the nonlinear relation $j_- = \alpha\, j_+$

Discrete Jumps in the BPS Spectrum

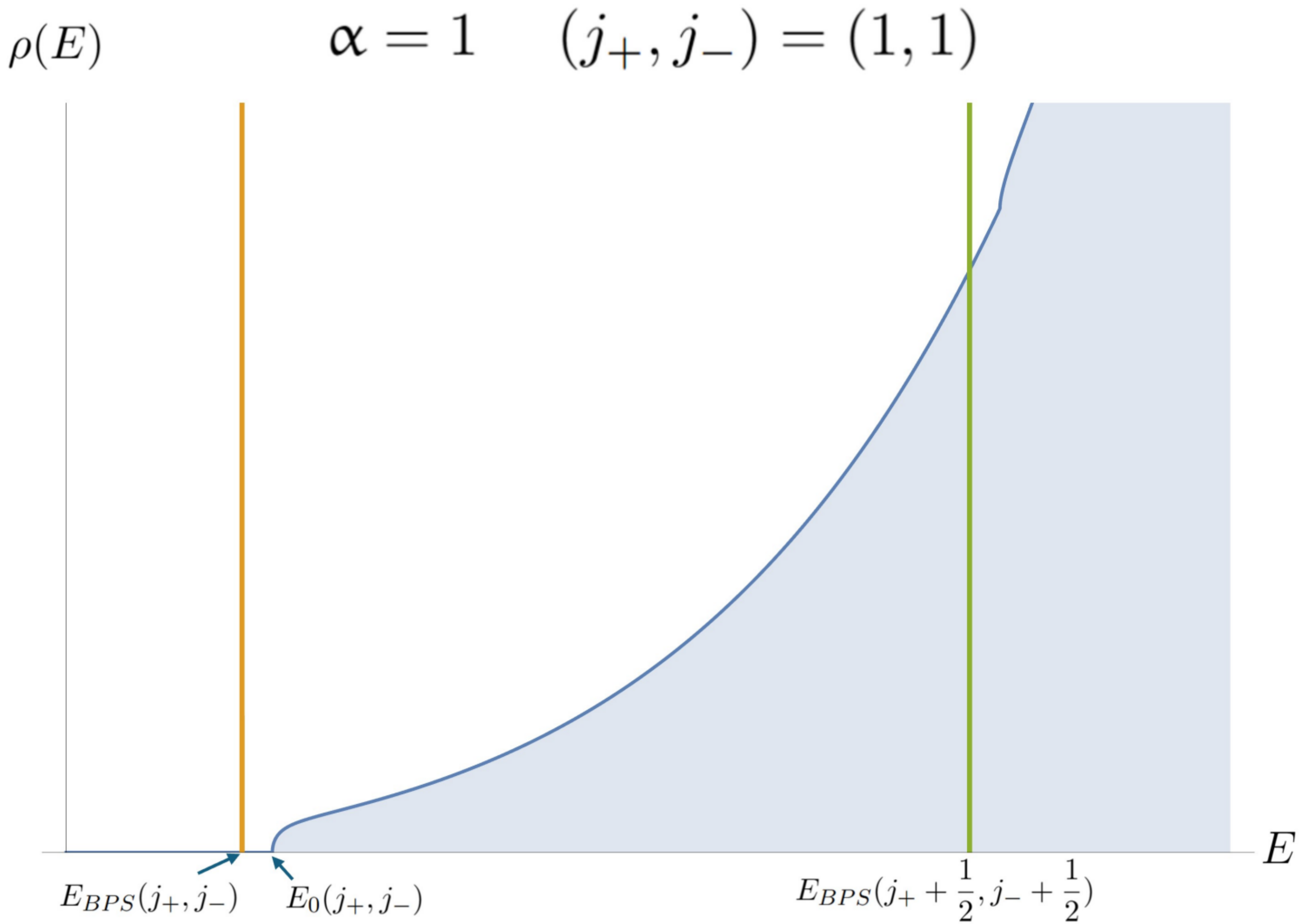


(a) NS sector



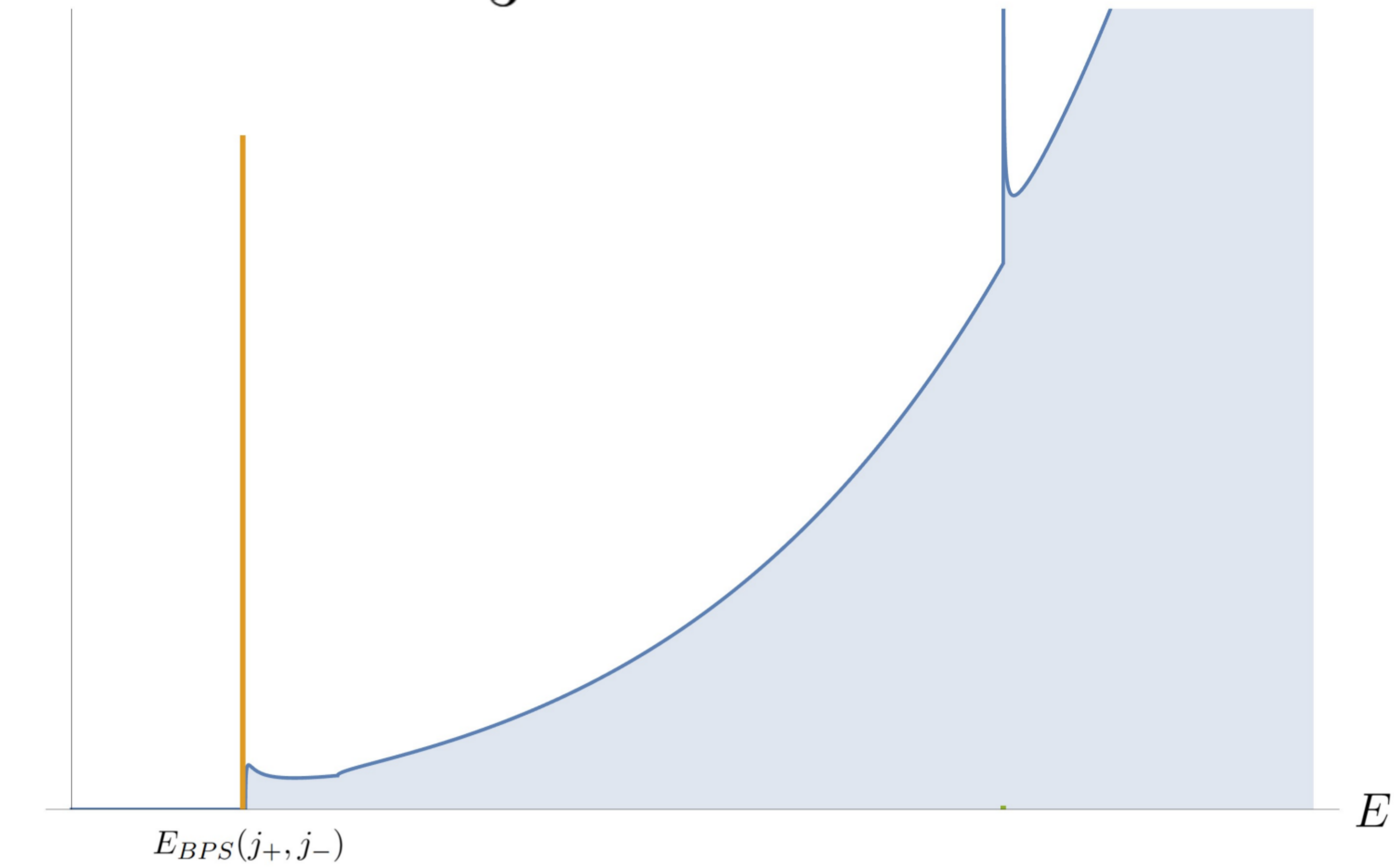
(b) R sector

Non-BPS Spectrum



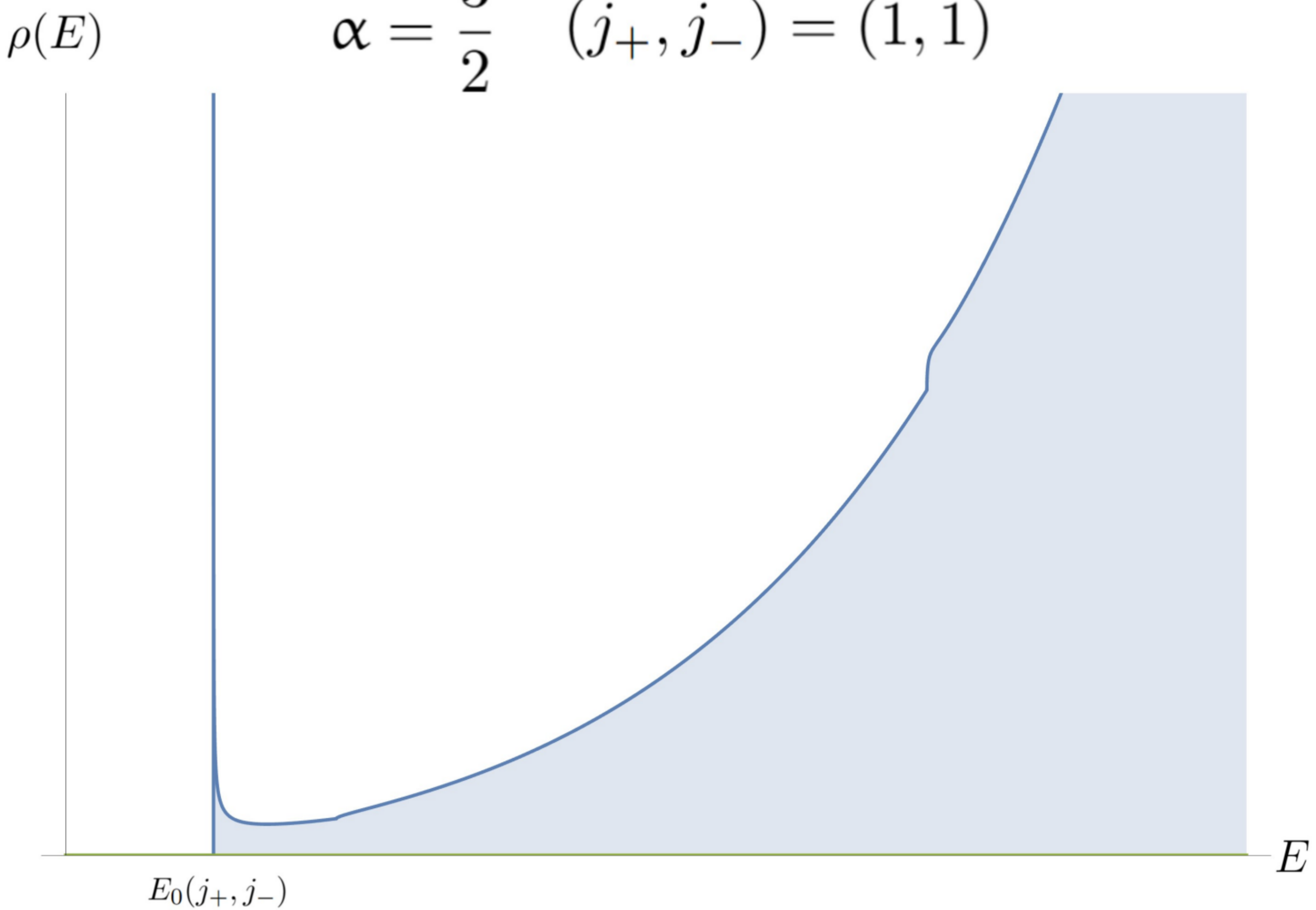
Non-BPS Spectrum

$$\alpha = \frac{4}{3} \quad (j_+, j_-) = (1, 1)$$



Non-BPS Spectrum

$$\alpha = \frac{3}{2} \quad (j_+, j_-) = (1,1)$$



Elliptic Genus from JT

If we only care about the BPS spectrum which could be compared with the (unknown) CFT calculation, we can compute the (modified) elliptic genus in JT directly. This is the same as taking an imaginary angular velocity:

$$\Omega_L^- = \Omega_L^+ + \frac{2\pi \mathrm{i}}{\beta}$$

The result is actually a temperature dependent index (due to the shifts of ground state energies of BPS superselection sectors):

$$\mathrm{Index}(\sigma) = \sum_{j\geq 0} \sum_{u\in u_0\cdot\mathbb{Z}} e^{2\pi \mathrm{i}\mu u} \sin 2\pi\sigma\, \chi_j(\sigma)$$
$$e^{S_0} \frac{2\sqrt{\pi\alpha}}{(1+\alpha)} \sin\left(\frac{\pi(2j+1)}{1+\alpha}\right) e^{-\beta\frac{\alpha}{2\Phi_r(1+\alpha)^2}((j+1/2)^2+u^2)}$$

Complex Metric with Non-holomorphic Action

In the interest of time, we merely state that the JT result can be reproduced directly from a 10d solution with imaginary angular velocity. The complex solution has an action that is non-holomorphic in the modular parameter. This once again implies a temperature dependent index!

$$I_{\text{on-shell}} = -\frac{\mathrm{i}\pi c}{12\tau} + \frac{2\pi \mathrm{i} k^+}{\tau}(\alpha_+ + n)^2 + \frac{2\pi \mathrm{i} k^-}{\tau}(\alpha_+ + n \pm 1/2)^2 + \text{h.c.}$$

A difficult calculation reveals our solution of Type IIB possesses an N=2 subalgebra of Killing spinors which are temperature dependent.

Generalization of [Coussaert, Henneaux '93; Giribet, Miskovic, Yazbek, Zanelli '24]

$$\begin{aligned}\varepsilon^i =& \frac{e^{\frac{(r_+ + r_-)}{2\ell}(t-\varphi)}}{\sqrt{\ell r}}\left(\sqrt{(r+r_+)(r+r_-)}\,\chi_{++-} + \sqrt{(r-r_+)(r-r_-)}\,\chi_{-+-}\right)\eta_1^i \\ &+ \frac{e^{-\frac{(r_+ + r_-)}{2\ell}(t-\varphi)}}{\sqrt{\ell r}}\left(\sqrt{(r-r_+)(r-r_-)}\,\chi_{+-+} + \sqrt{(r+r_+)(r+r_-)}\,\chi_{--+}\right)\eta_2^i\end{aligned}$$

Discussion

Our results using the near extreme limit give constraints on how much susy black holes can preserve, consistent with but not relying on string theory.

With large amounts of supersymmetry, the gravitational path integral contains fine grained, genuinely quantum gravity information about the spectrum.

Even for BPS observables, our method makes new predictions which have no known microscopic derivation.

If one had a weakly interacting dual SCFT, we could probe questions related to:

- BPS chaos [Lin, Maldacena, Rozenberg, Shan ‘22; Y. Chen, Lin, Shenker ‘24]
- Random Matrix Theory [Turiaci, Witten ‘23, Johnson, Kolanowski ’25]
- Black Hole Cohomologies and Fortuity [Choi, S. Kim, E. Lee ‘22; Chang, Y. Lin ’24; Chang, Y. Chen, Sia, Z. Yang ’24; Hughes, Shigemori ‘25]
- Microscopic computation of the temperature dependent index?

Bonus: $AdS_2 \times S^2 \times S^2 \times T^5$ and M-branes

	0	1	2	3	4	5	6	7	8	9	10
$M2_1$	—	—	—								
$M2_2$	—			—	—						
$M5_1$	—		—		—	—	—	—			
$M5_2$	—	—		—		—	—	—			
$M5_3$	—		—	—					—	—	—
$M5_4$	—	—			—				—	—	—

$$S_0 = \frac{(2\pi)^7 R_+^2 R_-^2 L^5}{G_N^{11}} = \frac{8 R_+^2 R_-^2 L^5}{\ell_P^9}$$

$$\alpha = \frac{R_-^2}{R_+^2}$$

To compute the JT gravity coupling, need the non-extremal solution!

Is there a asymptotically flat 6d or 7d black hole with this horizon topology?

$$ds^2 = -r^2\left(\frac{1}{R_+^2} + \frac{1}{R_-^2}\right)\mathrm{d}t^2 + \left(\frac{1}{R_+^2} + \frac{1}{R_-^2}\right)^{-1}\frac{\mathrm{d}r^2}{r^2}$$
$$+ R_+^2(\mathrm{d}\theta_+^2 + \sin^2(\theta_+)\mathrm{d}\phi_+^2) + R_-^2(\mathrm{d}\theta_-^2 + \sin^2(\theta_-)\mathrm{d}\phi_-^2) + L^2\mathrm{d}\theta_i\mathrm{d}\theta^i\,,$$

$$F_4 = L^2\Big(\frac{\sqrt{R_+^2 + R_-^2}}{R_+R_-}\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2 + \frac{\sqrt{R_+^2 + R_-^2}}{R_+R_-}\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\theta_4$$
$$+ R_+\sin(\theta_+)\mathrm{d}\theta_+\wedge\mathrm{d}\phi_+\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta_3 - R_+\sin(\theta_+)\mathrm{d}\theta_+\wedge\mathrm{d}\phi_+\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\theta_4$$
$$+ R_-\sin(\theta_-)\mathrm{d}\theta_-\wedge\mathrm{d}\phi_-\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta_4 + R_-\sin(\theta_-)\mathrm{d}\theta_-\wedge\mathrm{d}\phi_-\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\theta_3\Big)\,,$$