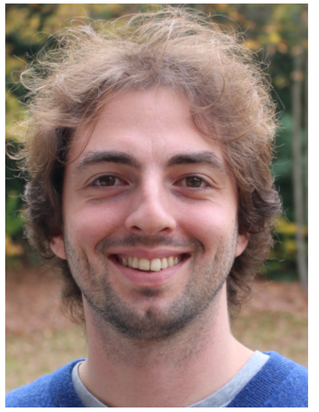
Mysterious properties of holographic entanglement

Matthew Headrick (Brandeis & IHES)

Rencontres Théoriciennes, 4 Dec 2025


Brianna G.-W.


Guglielmo G.

Based on:
- 2407.07165 by B. Grado-White, G. Grimaldi, M.H., V. Hubeny
- 2508.21823 by G. Grimaldi, M.H., V. Hubeny
- 2512.????? by G. Grimaldi, M.H., V. Hubeny, P. Shteyner


Veronika H.

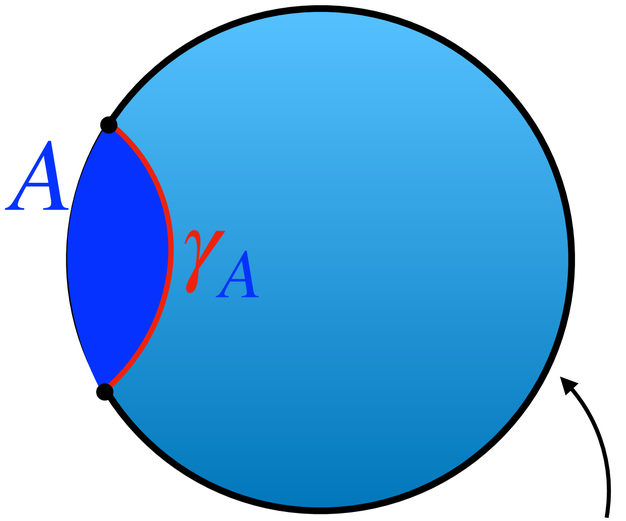

Pavel S.

The RT formula [Ryu-Takayanagi '06] (& generalizations) is a cornerstone of our modern understanding of *holography* & *quantum gravity*:

- tensor network models
- bulk reconstruction
- error correction
- unitary black-hole evaporation
- complexity
- …

(as well as QFT & many-body physics)

$$S_A = \min_{\gamma \sim A} \frac{|\gamma|}{4G_{\mathrm{N}}}$$



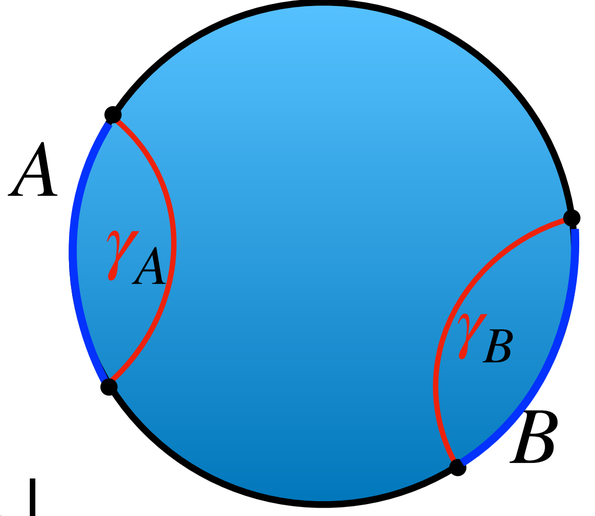
(instantaneously) static slice

$$\left[\; S_A := -\operatorname{Tr}(\rho_A \ln \rho_A)\,, \quad \rho_A := \operatorname{Tr}_{A^c} \rho \;\right]$$

RT still has much to teach us about holography, quantum gravity, & general relativity (+ QFT)

This talk is about *entropy inequalities* obeyed by RT

Entropy inequalities obeyed by RT



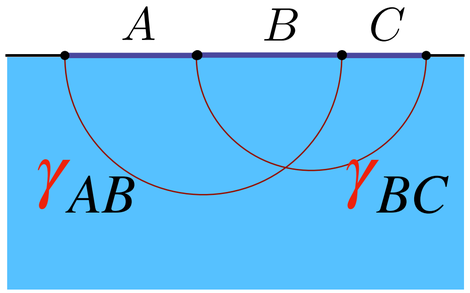
1) Subadditivity: $\boxed{S_A + S_B \geq S_{AB}}$ or $\boxed{I_{A:B} \geq 0}$

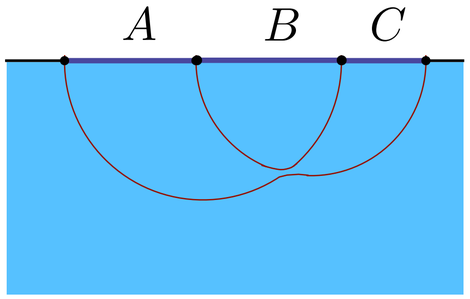
[mutual information $I_{A:B} := S_A + S_B - S_{AB}$]

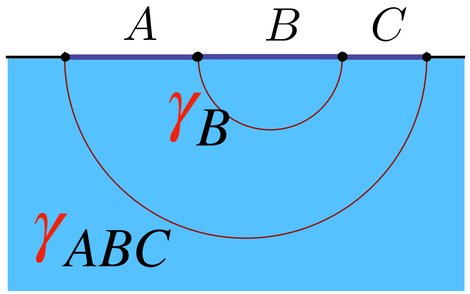
Proof: $\gamma_a \gamma_B \sim AB \;\Rightarrow\; S_{AB} = \min_{\gamma \sim AB} \frac{|\gamma|}{4G_N} \leq \frac{|\gamma_A \gamma_B|}{4G_N} = \frac{|\gamma_A| + |\gamma_B|}{4G_N} = S_A + S_B$

2) Strong subadditivity (SSA): $\boxed{S_{AB} + S_{BC} \geq S_B + S_{ABC}}$ or $\boxed{I_{A:BC} \geq I_{A:B}}$

Proof: [MH-Takayanagi ’07, MH ’13]

$S_{AB} + S_{BC} =$ 

$=$ 

$\geq$ 

$= S_{ABC} + S_B$

Subadditivity & SSA are general properties of quantum states

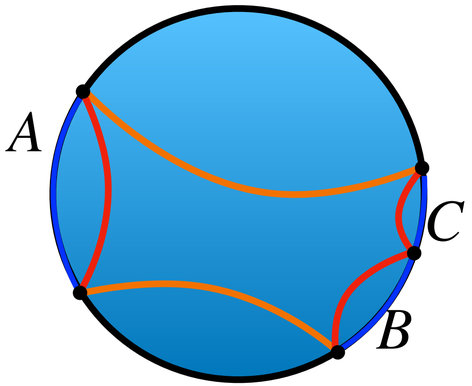
Entropy inequalities obeyed by RT

3) MMI [Hayden-MH-Maloney '11, MH '13]: $S_{AB} + S_{BC} + S_{AC} \geq S_A + S_B + S_C + S_{ABC}$

or $I_{A:BC} \geq I_{A:B} + I_{A:C}$

Same kind of cut-paste proof

Example:



$I_{A:B} = I_{A:C} = 0\,, \quad I_{A:BC} > 0$

like perfect tensor or random state (Page curve)

Not a general property of quantum states

Counterexample: $|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$

Special property of holographic states, at classical level

(can be violated by quantum corrections)

Possible explanation: "bipartite dominance conjecture" [Cui-Hayden-He-MH-Stoica-Walter '18]

Entropy inequalities obeyed by RT

Systematic search initiated by [Bao-Nezami-Ooguri-Stoica-Sully-Walter '15]

Contraction map:

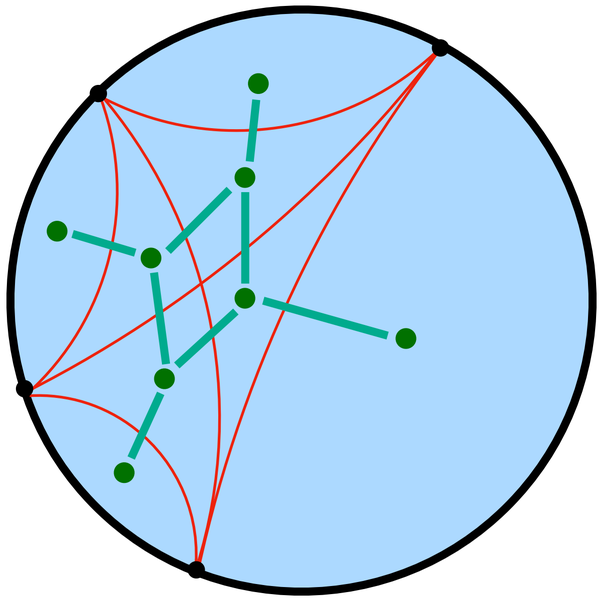
- Systematize cut-paste proof method
- Computer searches for inequalities

Holographic (RT) entropy cone:

- Set of entropy vectors $(S_A, S_B, S_{AB}, \ldots) \in \mathbb{R}^{2^N-1}$ allowed for N regions
- Convex polyhedral cone

Graph model:

- Decompose bulk along all RT surfaces $\gamma_A, \gamma_B, \gamma_{AB}, \ldots$
- Vertex = bulk cell
- Edge = partial surface, weight = area$/4G_N$
- *On resulting weighted graph, entropies are min cuts*



Conversely, for any weighted graph there is a holographic state

$\Rightarrow$ finding RT inequalities is a problem in graph theory

Entropy inequalities obeyed by RT

Results so far:

- No additional inequalities for $N \leq 4$
- 5 additional inequalities for $N = 5$ [Hernández Cuenca '19]:

1. $S_A + S_B \geq S_{AB}$

2. $S_{AB} + S_{AC} + S_{BC} \geq S_A + S_B + S_C + S_{ABC}$

3. $S_{ABC} + S_{ADE} + S_{BCDE} \geq S_A + S_{BC} + S_{DE} + S_{ABCDE}$

4. $S_{ABC} + S_{ABD} + S_{ACE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{BD} + S_{CE} + S_{DE} + S_{ABCDE}$

5. $S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ACE} + S_{ADE} + S_{BCE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BE} + S_{CE} + S_{DE} + S_{BCD} + S_{ABCE} + S_{ABDE} + S_{ACDE}$

6. $3S_{ABC} + 3S_{ABD} + S_{ABE} + S_{ACD} + 3S_{ACE} + S_{ADE} + S_{BCD} + S_{BCE} + S_{BDE} + S_{CDE} \geq 2S_{AB} + 2S_{AC} + S_{AD} + S_{AE} + S_{BC} + 2S_{BD} + 2S_{CE} + S_{DE} + 2S_{ABCD} + 2S_{ABCE} + S_{ABDE} + S_{ACDE}$

7. $2S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ADE} + S_{BCE} + S_{BDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BC} + S_{BE} + S_{DE} + S_{ABCD} + S_{ABCE} + S_{ABDE}$

8. $S_{AD} + S_{BC} + S_{ABE} + S_{ACE} + S_{ADE} + S_{BDE} + S_{CDE} \geq S_A + S_B + S_C + S_D + S_{AE} + S_{DE} + S_{BCE} + S_{ABDE} + S_{ACDE}$

- 1800 known additional inequalities for $N = 6$ [Hernández Cuenca-Hubeny-Jia '23]
- Two known infinite families [Czech-Liu-Yu '24]

Entropy inequalities obeyed by RT

Questions about these inequalities:

1. Is there a general structure or pattern to them?

   [He-MH-Hubeny ’19, He-Hubeny-Rangamani ’20, Hernández Cuenca-Hubeny-Rota ’22, Hernández Cuenca-Hubeny-Jia ’23]

2. What do they tell us about the nature of entanglement in semiclassical states in quantum gravity?
   Is there an ansatz for such states that automatically obeys them?

   [Cui-Hayden-He-MH-Stoica-Walter ’18, Czech-Shuai-Wang-Zhang ’23, Czech-Shuai ’25,Czech-Shuai-Wang ‘25]

3. Are they valid beyond the Einstein gravity (strong-coupling) limit?

4. Are they valid in non-static states?

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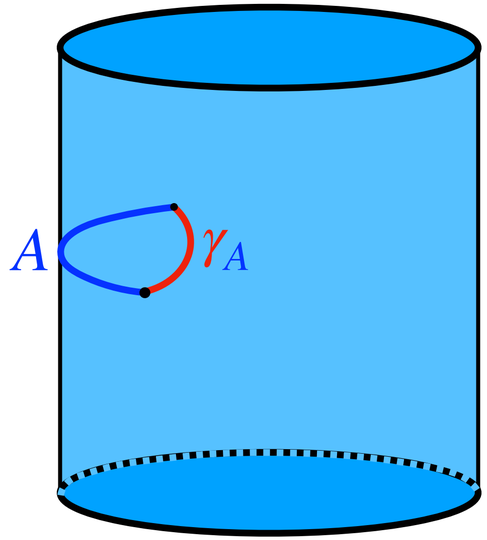
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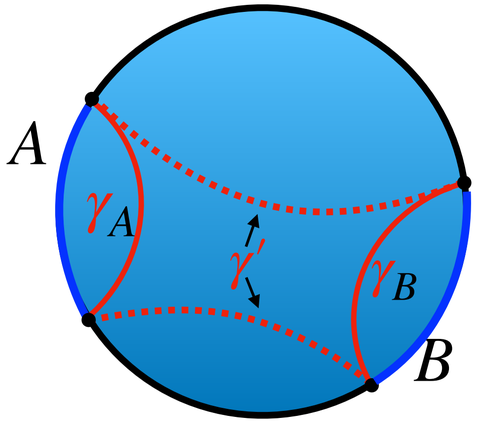
Are RT entropy inequalities obeyed by HRT?

HRT formula [Hubeny-Rangamani-Takayanagi '07]:

$$S_A = \min_{\substack{\gamma \sim A \\ \text{extremal}}} \frac{|\gamma|}{4G_N}$$



$A, B, \ldots$ are disjoint boundary regions on the same Cauchy slice

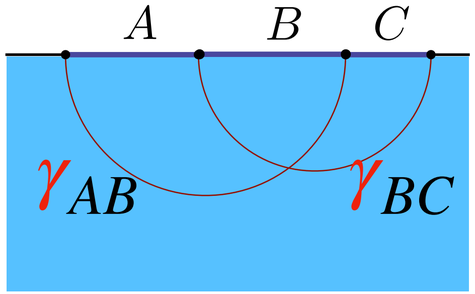


Subadditivity: same proof

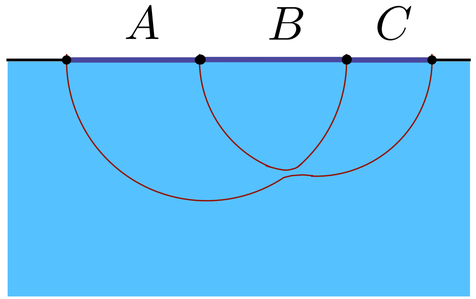
$$S_{AB} = \min\left\{ \frac{|\gamma_A| + |\gamma_B|}{4G_N}, \frac{|\gamma'|}{4G_N} \right\} \leq \frac{|\gamma_A| + |\gamma_B|}{4G_N} = S_A + S_B$$

SSA: Old proof fails since γ_{AB} & γ_{BC} do not generically lie on same slice

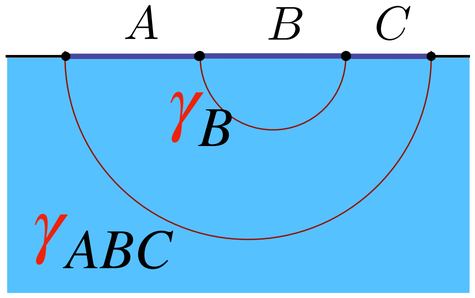
$$S_{AB} + S_{BC} =$$



$=$



$\geq$

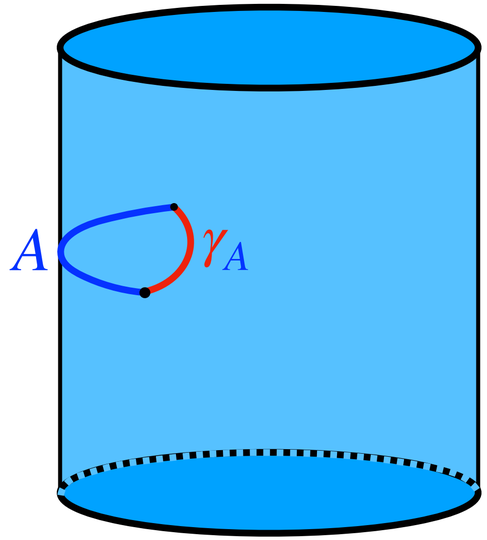


$= S_{ABC} + S_B$

Are RT entropy inequalities obeyed by HRT?

HRT formula [Hubeny-Rangamani-Takayanagi '07]:

$$S_A = \min_{\substack{\gamma \sim A \\ \text{extremal}}} \frac{|\gamma|}{4G_N}$$



Two proofs that HRT obeys SSA:

- maximin [Wall '14]
- direct [Bousso-Kaya '24]

Both proofs require:

- Einstein equation
- Null energy condition
- AdS boundary conditions
- Global hyperbolicity

$\Rightarrow$ focusing, $\dot{\theta} \leq 0$

($\theta :=$ expansion of null congruence)

These are general relativity theorems

Are RT entropy inequalities obeyed by HRT?

What about the *special* RT inequalities?

Evidence that they are obeyed, in several independent regimes:

- MMI *proved*, using same techniques as strong subadditivity [Wall ’14, Bousso-Kaya ’24]
- All inequalities *proved* in scaling limit of late times & large distances, using membrane theory of entanglement dynamics [Bao-Mezei ’18]
- For 2+1 bulk:
  - All inequalities *proved* when bulk is simply connected [Czech-Dong ’19]
  - All inequalities *proved* for 2-boundary, genus-0 wormholes [Grado-White-Grimaldi-MH-Hubeny ’25]
  - All known inequalities up to 6 regions (& a few for more than 6 regions) *checked numerically* on thousands of randomly generated vacuum multiboundary wormholes [Grado-White-Grimaldi-MH-Hubeny ’25]

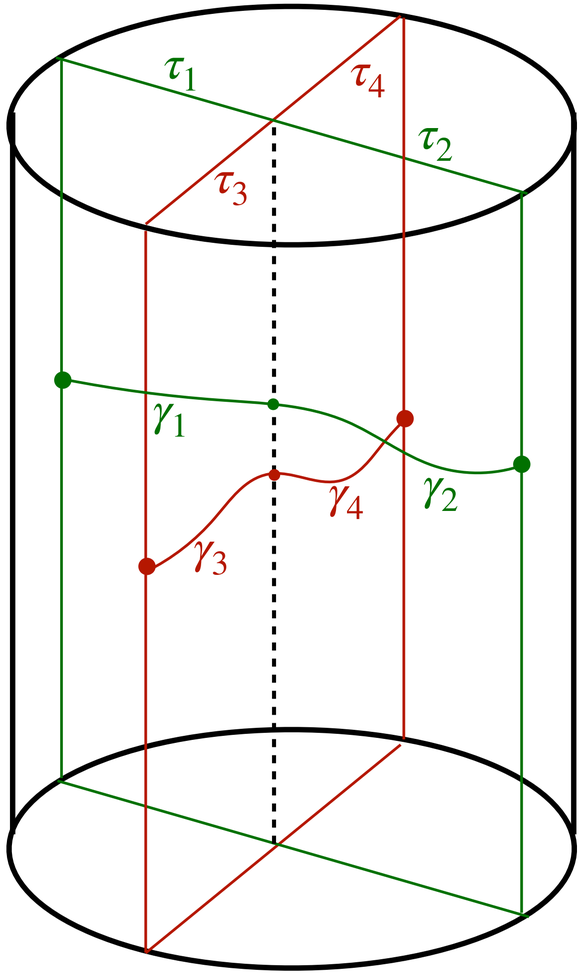
Are RT entropy inequalities obeyed by HRT?

A beautiful problem in GR

An attempt to prove that HRT obeys *all* RT inequalities

[Grado-White-Grimaldi-MH-Hubeny ’25]:

- Put HRT surfaces on *time-sheets* (minimax)
- Construct graph model where vertices are *spacetime* regions, such that computing entropies reduces to finding min cuts



Are RT entropy inequalities obeyed by HRT?

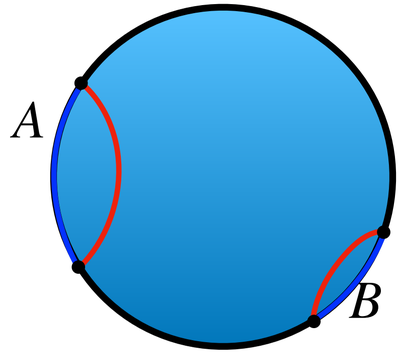
Suppose someone told you there exists a counterexample:

Bulk spacetime & boundary regions whose entropies violate some RT inequality

How would you find it?

One strategy:

- Start with a configuration that *saturates* some inequality
- Try to deform the bulk to *violate* it



$\gamma_{AB} = \gamma_A \cup \gamma_B$

Two classes of saturating configurations:

1. *Same* HRT surfaces on two sides

   Deformation changes LHS & RHS by the same amount ⇒ no counterexample :(

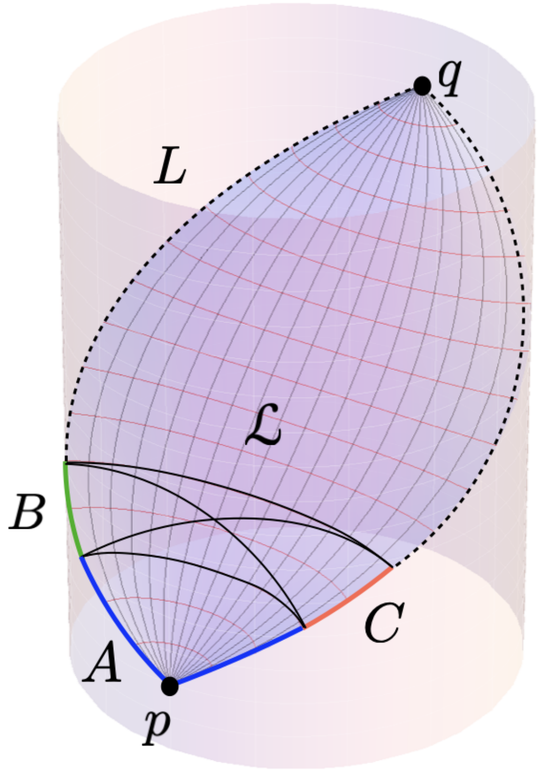
2. *Different* HRT surfaces, same total area

   Deformation changes LHS & RHS differently ⇒ possible counterexample :)

Are RT entropy inequalities obeyed by HRT?

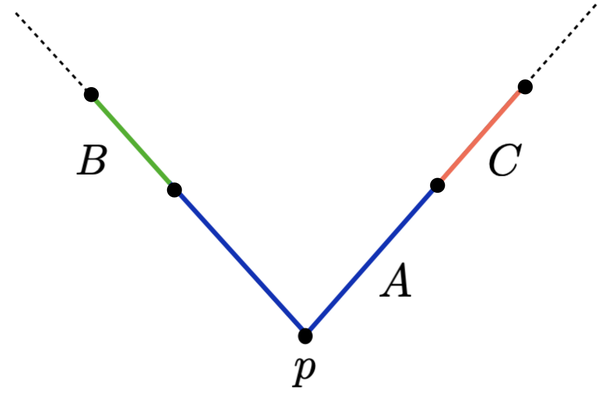
How to construct a configuration with different HRT surfaces but same total area?

Example: SSA $S_{AB} + S_{AC} \geq S_A + S_{ABC}$



State: vacuum (bulk = empty AdS)

Configuration: null trapezoid



SSA is saturated (Markov state):

- HRT surfaces lie on bulk null congruence from p
- Every geodesic passes through every HRT surface exactly once
- Congruence has expansion $\theta = 0 \Rightarrow$ area element is constant

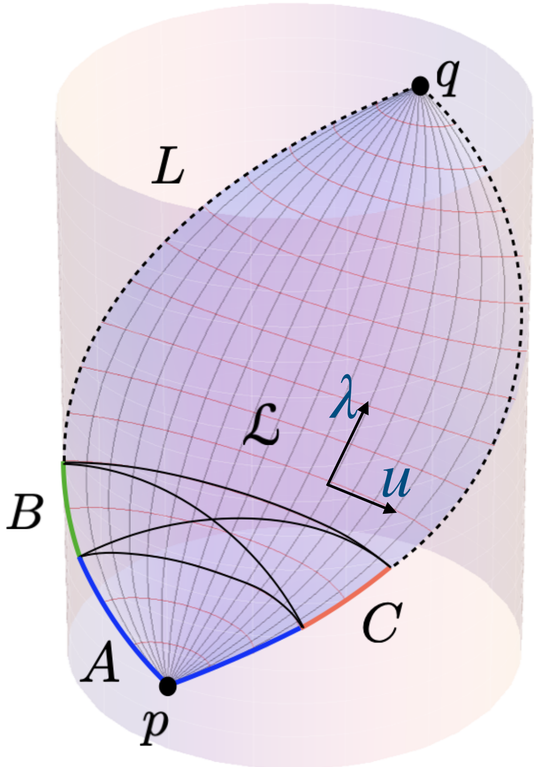
Now deform bulk metric to make S_{AB}, S_{AC} smaller & S_A, S_{ABC} larger

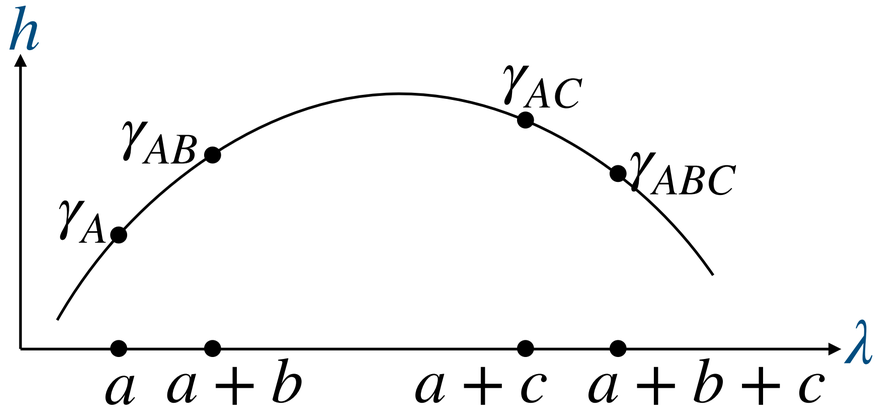
Oops, we just violated SSA!

Are RT entropy inequalities obeyed by HRT?

What did we forget? Deformation must obey *focusing:* $\dot{\theta} \leq 0$

$$\theta = \dot{h}\,, \quad h := \delta g_u{}^u \quad \Rightarrow \quad \ddot{h} \leq 0$$





HRT	$\lambda :=$ affine par.
γ_A	a
γ_{AB}	$a+b$
γ_{AC}	$a+c$
γ_{ABC}	$a+b+c$

a, b, c depend on geodesic $(b, c > 0)$

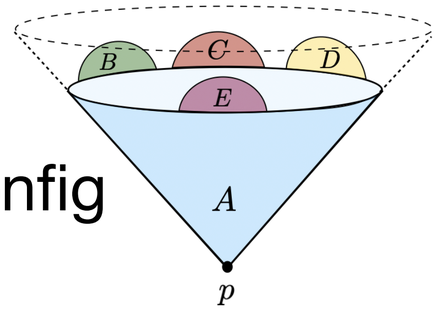
$$\ddot{h} \leq 0 \;\Rightarrow\; h(a+b) + h(a+c) \geq h(a) + h(a+b+c)$$

$$\Rightarrow\; \delta|\gamma_{AB}| + \delta|\gamma_{AC}| \geq \delta|\gamma_A| + \delta|\gamma_{ABC}|$$

(summing contributions from all geodesics)

SSA is saved!

Are RT entropy inequalities obeyed by HRT?



For an inequality with more regions, need 2+1 boundary for light-cone config

Example:

$$S_{ABC} + S_{BCD} + S_{CDE} + S_{ADE} + S_{ABE} \geq S_{AB} + S_{BC} + S_{CD} + S_{DE} + S_{AE} + S_{ABCDE}$$

Terms without A cancel in any state — "null reduction on A":

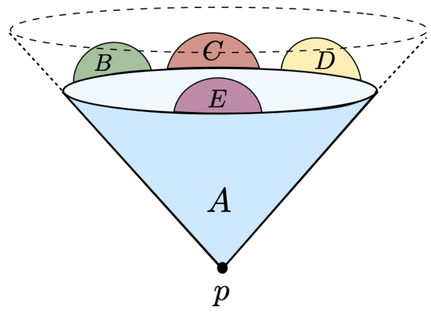
$$S_{ABC} + S_{ADE} + S_{ABE} \geq S_{AB} + S_{AE} + S_{ABCDE}$$

Reduced inequality saturated in vacuum

(generalized Markov state [Casini-Teste-Torroba '17])

Can we violate it by a bulk deformation?

Are RT entropy inequalities obeyed by HRT?



$$S_{ABC} + S_{ADE} + S_{ABE} \geq S_{AB} + S_{AE} + S_{ABCDE}$$

Reduced inequality saturated in vacuum

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Can we violate it by a bulk deformation?

Linearity of HRT PDE $\Rightarrow$

γ_{ABC} at $\lambda = a+b+c$, γ_{ADE} at $\lambda = a+d+e$, etc. with $b, c, d, e > 0$

Theorem: $\forall$ concave function h & $\forall\, b, c, d, e > 0$,

$$h(a+b+c) + h(a+d+e) + h(a+b+e) \geq h(a+b) + h(a+e) + h(a+b+c+d+e)$$

$$\Rightarrow \quad \delta|\gamma_{ABC}| + \delta|\gamma_{ADE}| + \delta|\gamma_{ABE}| \geq \delta|\gamma_{AB}| + \delta|\gamma_{AE}| + \delta|\gamma_{ABCDE}|$$

So this inequality is safe!

Majorization tutorial

Given $\vec{x}, \vec{y}$ such that $\sum_i x_i = \sum_i y_i$, the following are equivalent:

- $\vec{x} = D\vec{y}$ for some doubly stochastic matrix D
- For all $k = 1, \ldots n$,
  (sum of k largest components of $\vec{x}$) $\leq$ (sum of k largest components of $\vec{y}$)
- For any concave function h, $\sum_i h(x_i) \geq \sum_i h(y_i)$

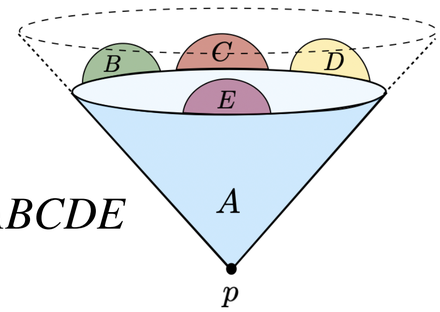
Write $\vec{x} \preceq \vec{y}$, say "$\vec{x}$ is majorized by $\vec{y}$"

Roughly, x_is are less spread out than y_is

Ex: $\vec{x} = (a+b+c, a+d+e, a+b+e)$, $\vec{y} = (a+b, a+e, a+b+c+d+e)$

$$(3,3,3) \preceq (2,2,5)$$

Are RT entropy inequalities obeyed by HRT?



$$S_{ABC} + S_{BCD} + S_{CDE} + S_{ADE} + S_{ABE} \geq S_{AB} + S_{BC} + S_{CD} + S_{DE} + S_{AE} + S_{ABCDE}$$

null reduce:

$$S_{ABC} + S_{ADE} + S_{ABE} \geq S_{AB} + S_{AE} + S_{ABCDE}$$

saturated in vacuum

vectorize:

$$\vec{x} := (a+b+c, a+d+e, a+b+e) \qquad \vec{y} := (a+b, a+e, a+b+c+d+e)$$

Since $\ddot{h} \leq 0$, inequality is safe iff $\vec{x} \preceq \vec{y} \;\; \forall\, b, c, \ldots > 0$

Theorem: C*ontraction map implies* $\vec{x} \preceq \vec{y} \;\; \forall\, b, c, \ldots > 0$

Conclusion:

RT inequalities *cannot* be violated by deformations of saturating configurations

Bonus: New combinatorial understanding of contraction maps

Are RT entropy inequalities obeyed by HRT?

Evidence that they are, in several independent regimes

- MMI *proved*, using same techniques as strong subadditivity [Wall '14, Bousso-Kaya '24]
- All inequalities *proved* in *scaling limit of late times & large distances,* using membrane theory of entanglement dynamics [Bao-Mezei '18]
- For 2+1 bulk:
  - All inequalities *proved* when bulk is simply connected [Czech-Dong '19]
  - All inequalities *proved* for 2-boundary, genus-0 wormholes [Grado-White-Grimaldi-MH-Hubeny '25]
  - All known inequalities up to 6 regions (& a few for more than 6 regions) *checked numerically* on thousands of randomly generated vacuum multiboundary wormholes [Grado-White-Grimaldi-MH-Hubeny '25]
- Cannot violate any inequality by deforming a saturating configuration [Grimaldi-MH-Hubeny '25, Grimalid-MH-Hubeny-Shteyner '25]

Entropy inequalities obeyed by RT

Exercises for the audience about these inequalities:

1. Is there a general structure or pattern to them?
2. What do they tell us about the nature of entanglement in semiclassical states in quantum gravity?
   Is there an ansatz for such states that automatically obeys them?
3. Are they valid beyond the Einstein gravity (strong-coupling) limit?
4. Are they valid in non-static states?

MERCI !