



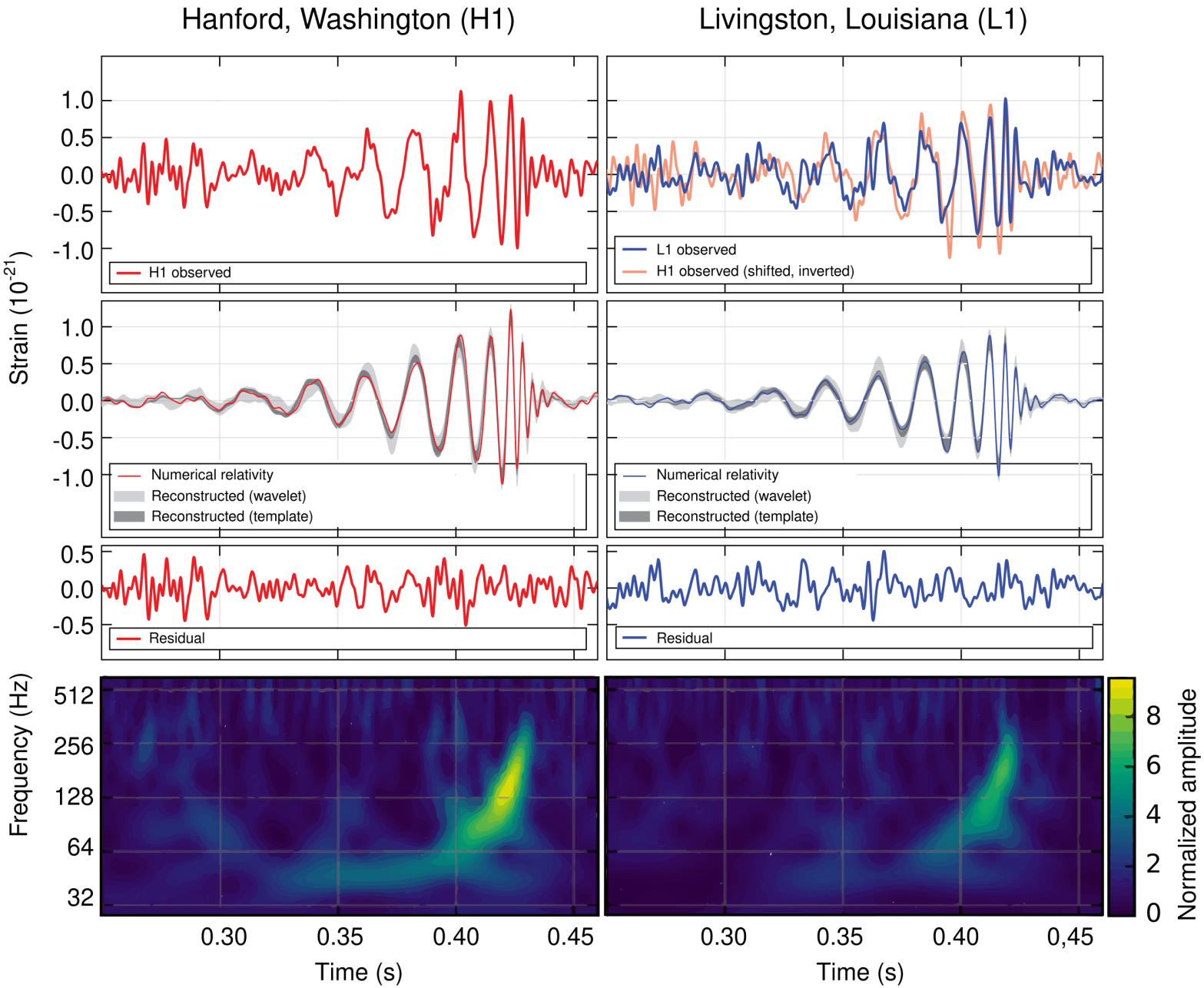
Hybrid waveforms at the crossroads between gravitational self force and post-Newtonian theory

David Trestini

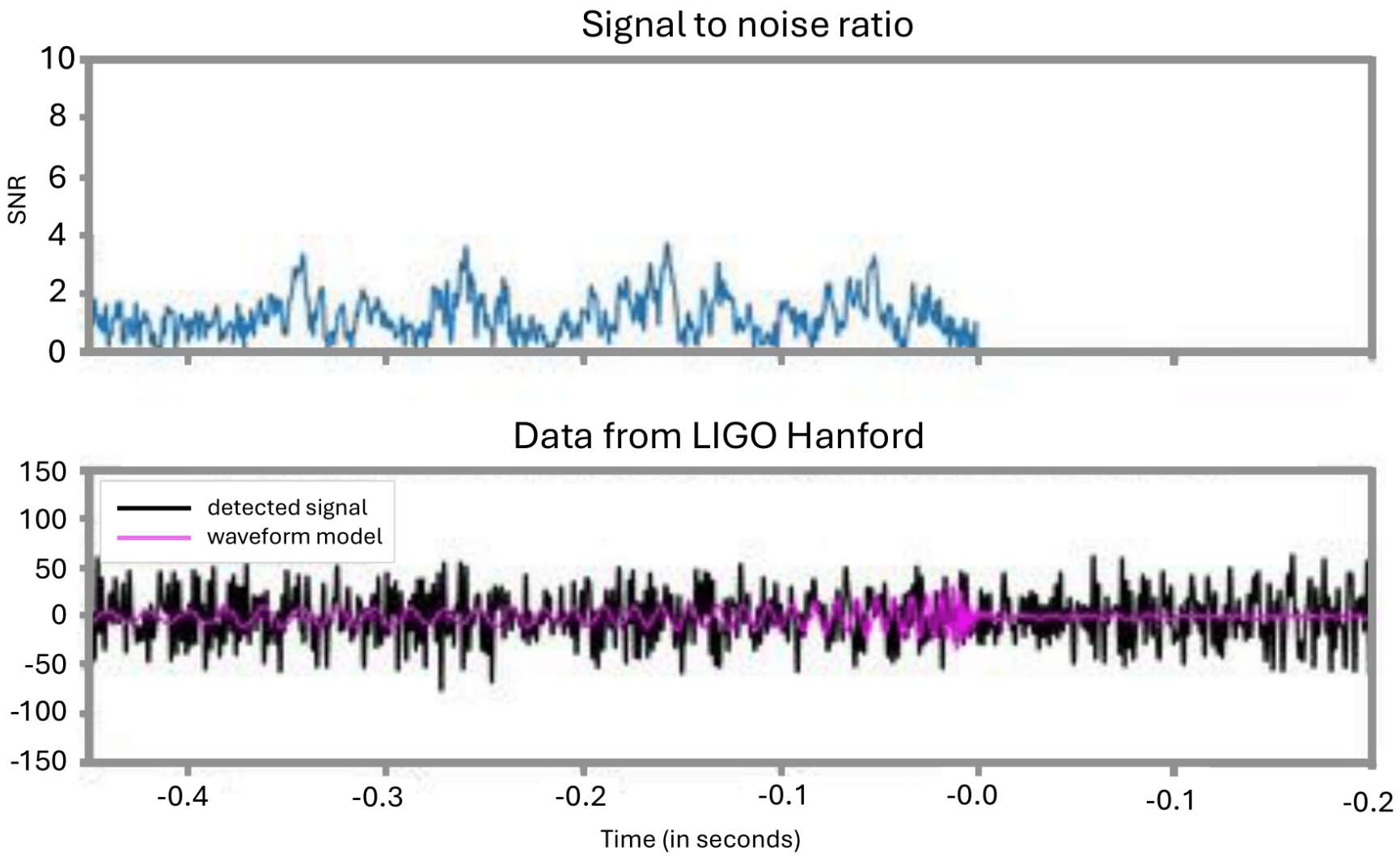
Monday, November 16, 2026

Institut d'Astrophysique de Paris — CNES

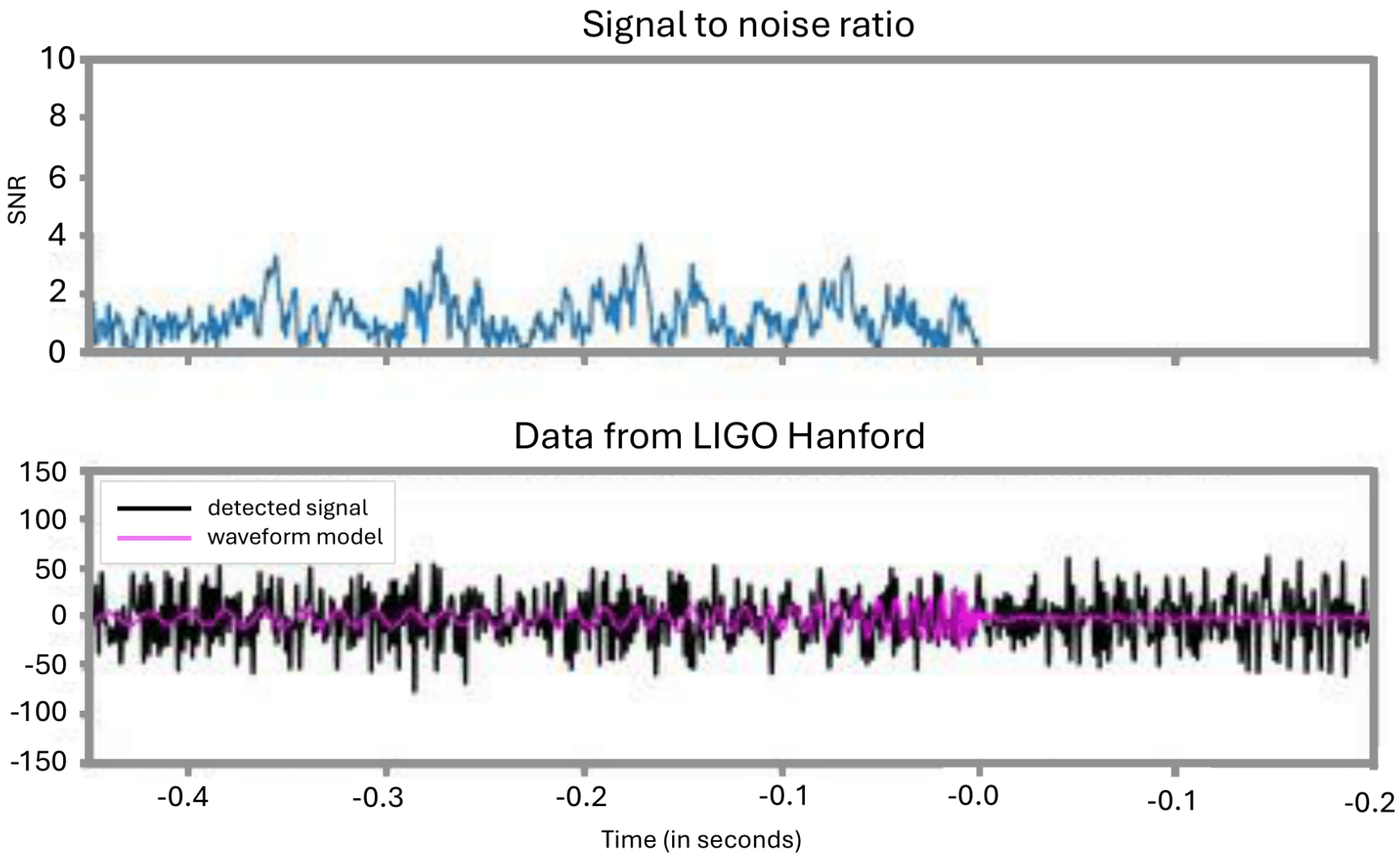
First detection: GW150914



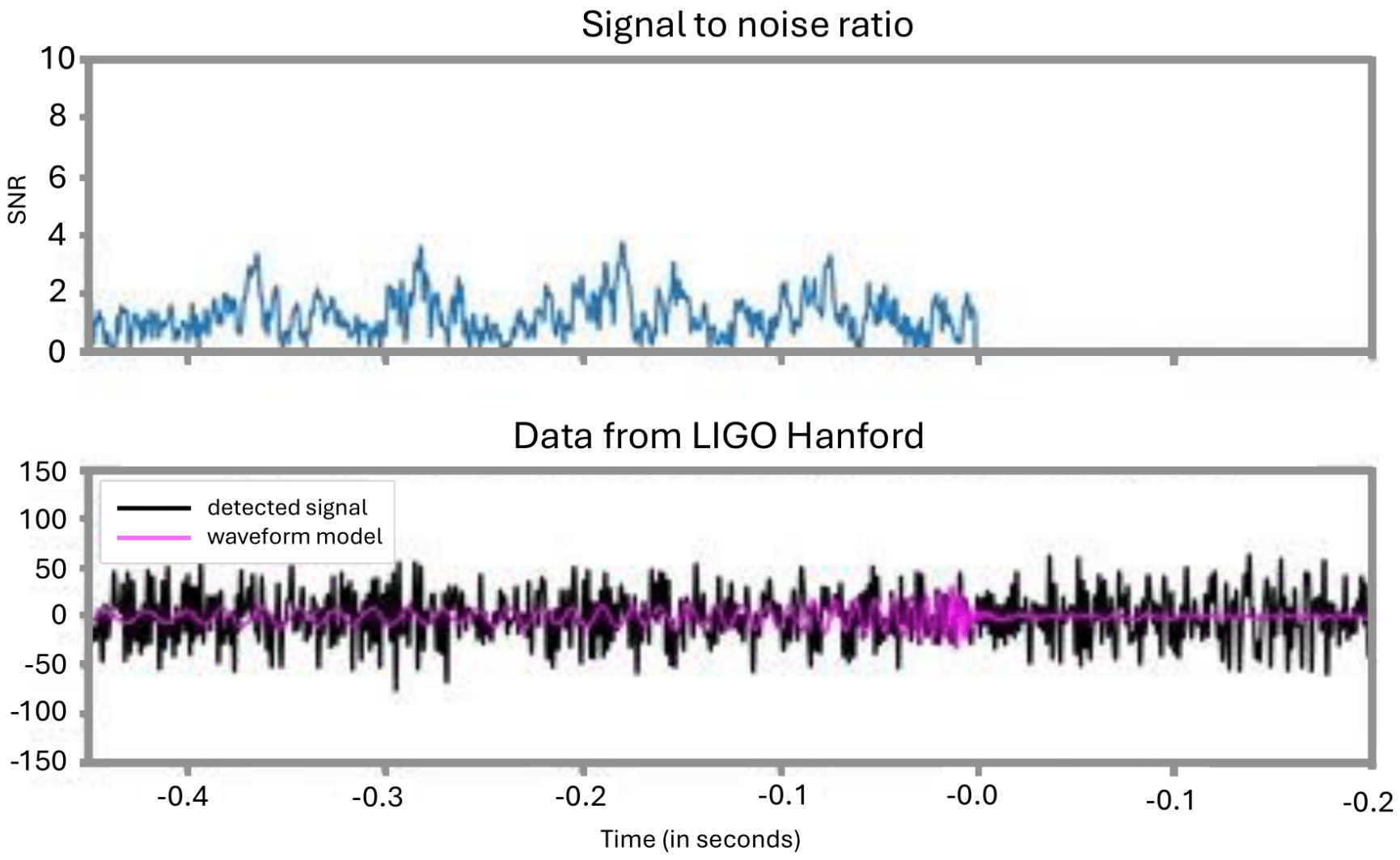
Matched filtering



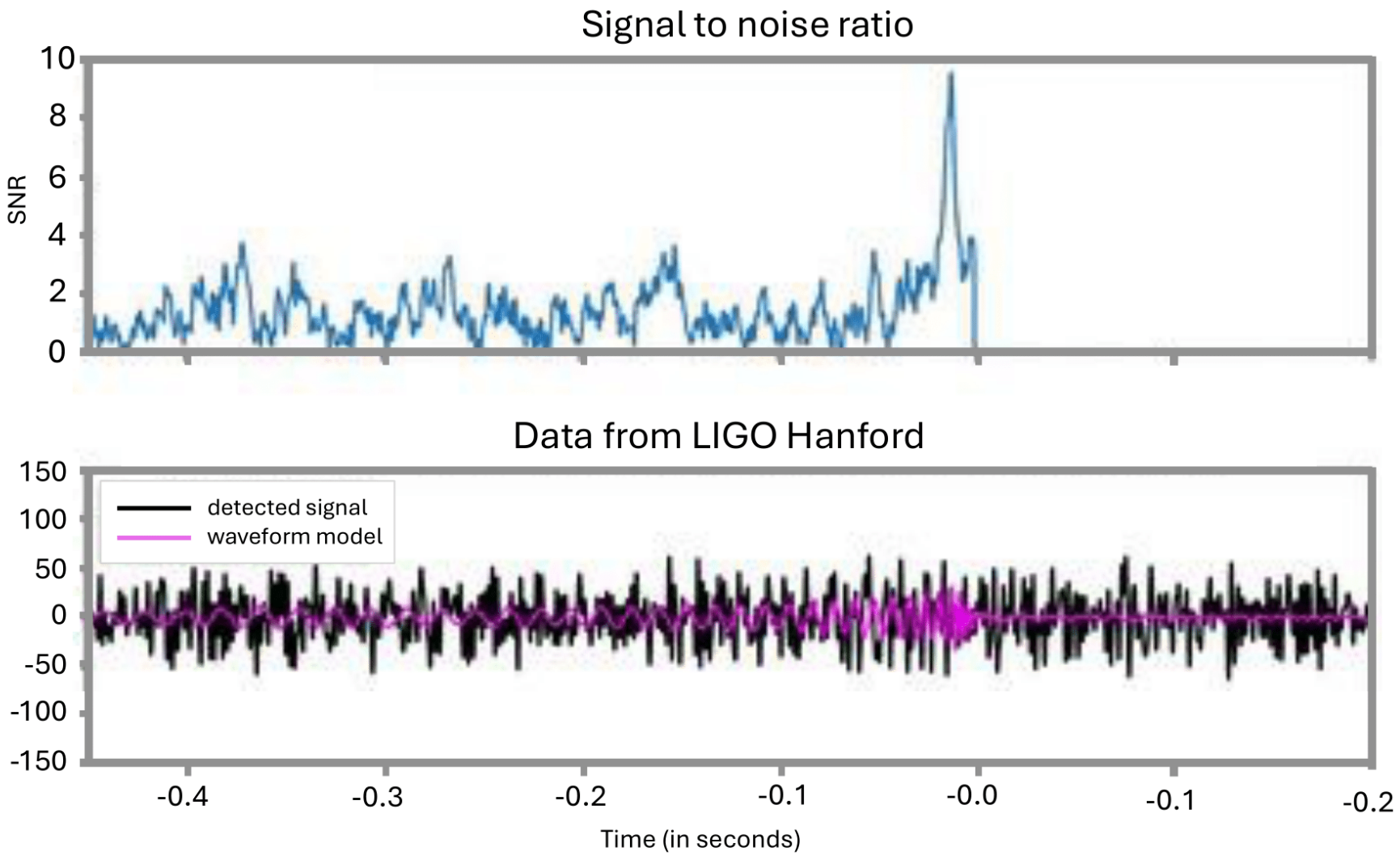
Matched filtering



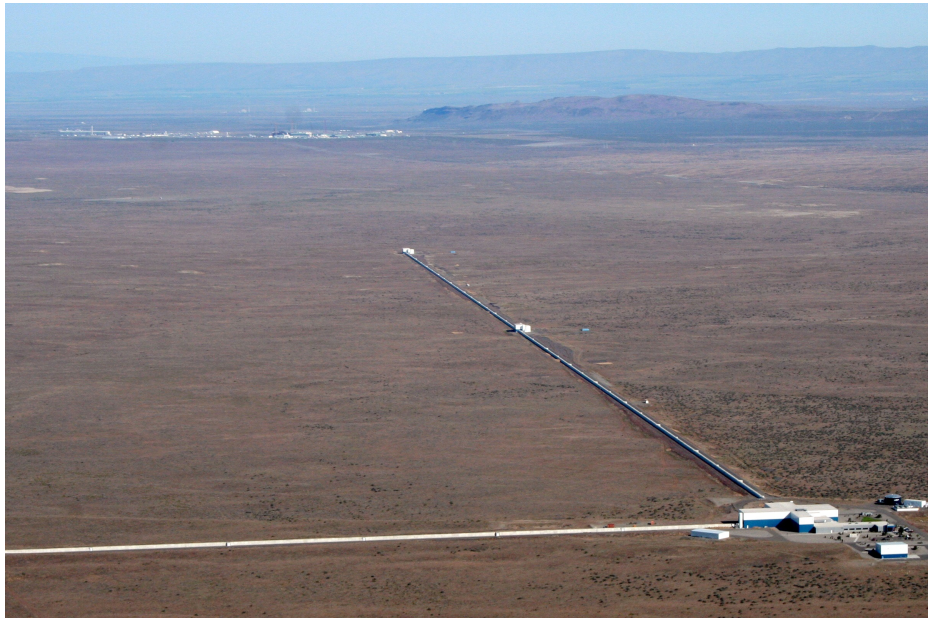
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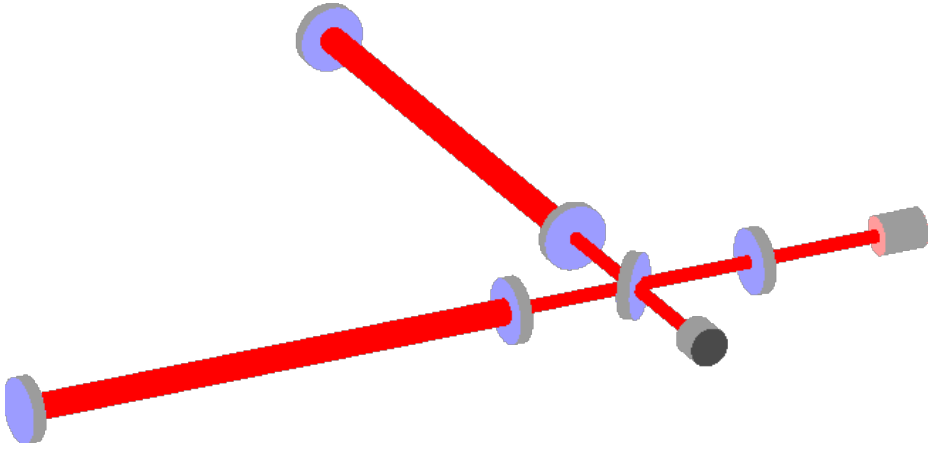


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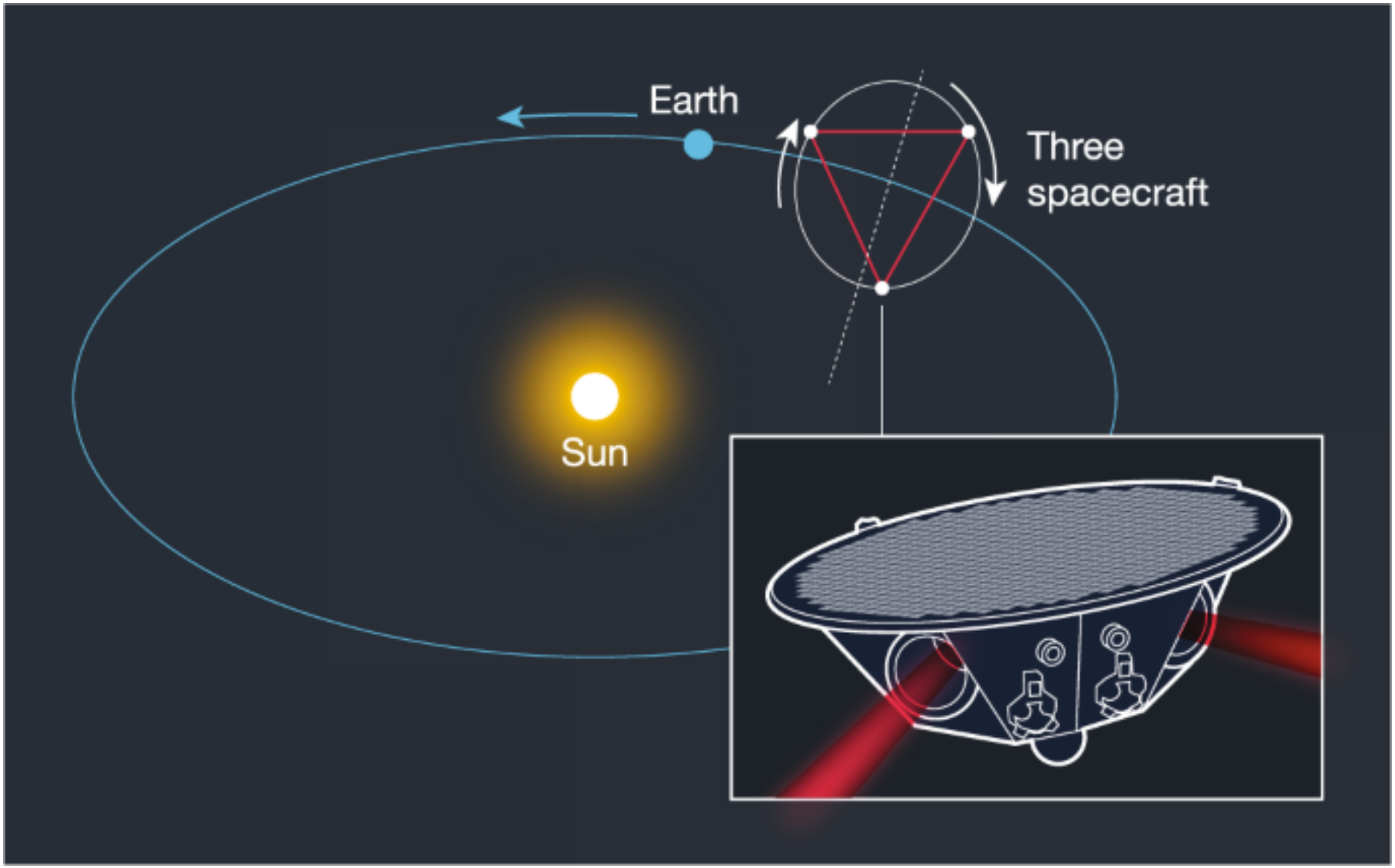


GW observatories: giant Michelson inteferometers

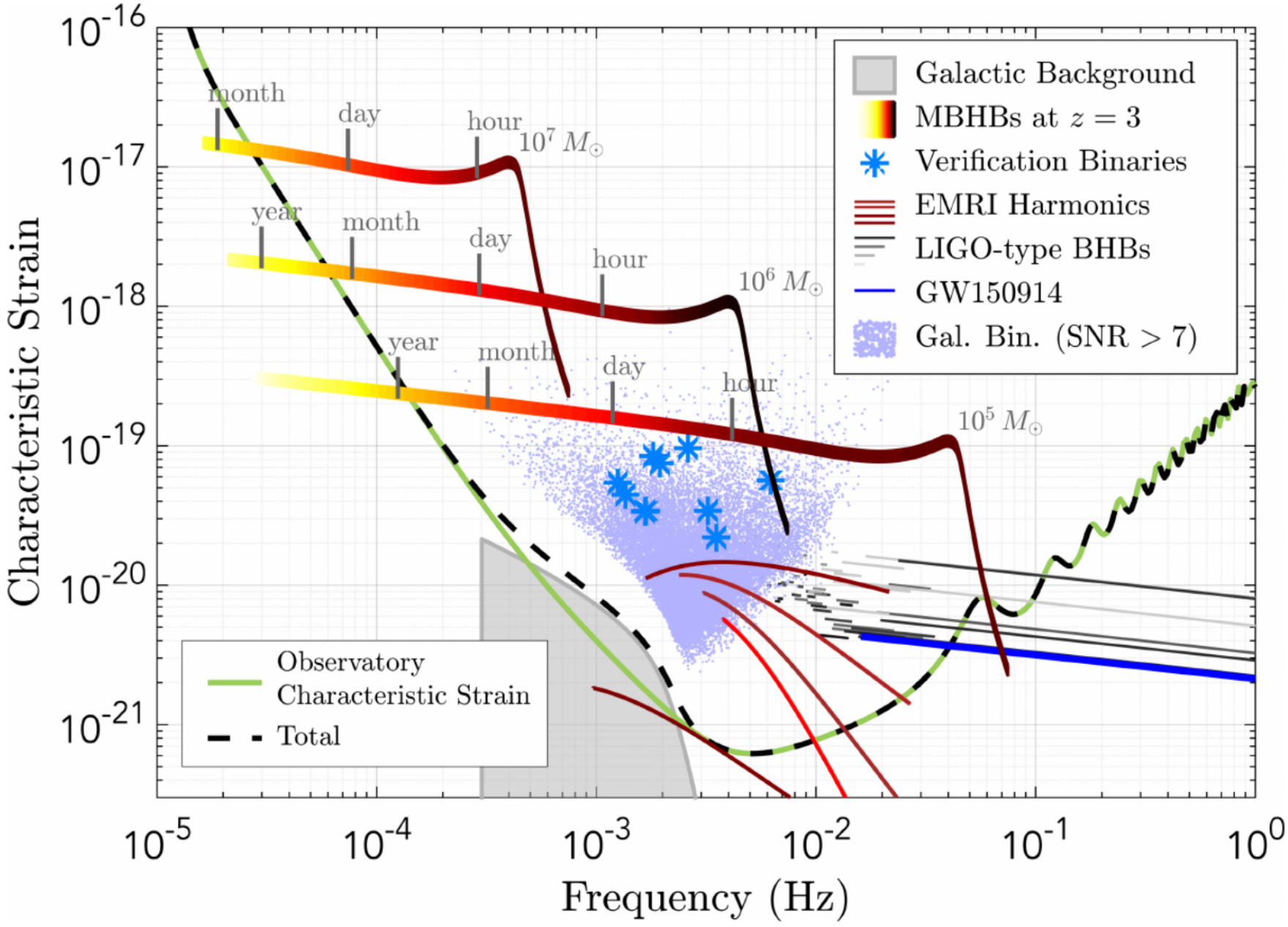




LISA



LISA sources



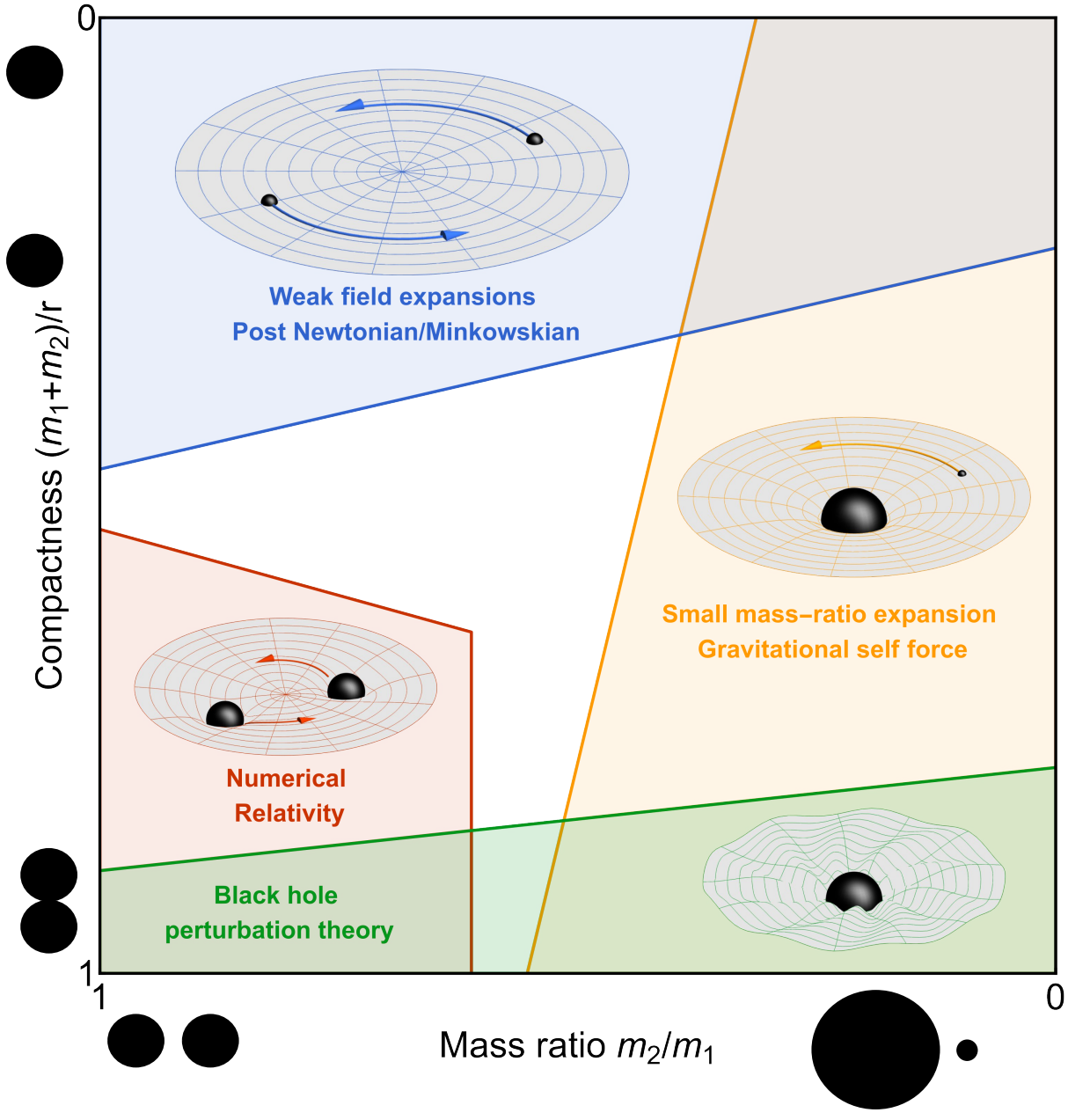
Bottom line

We need very accurate waveform models

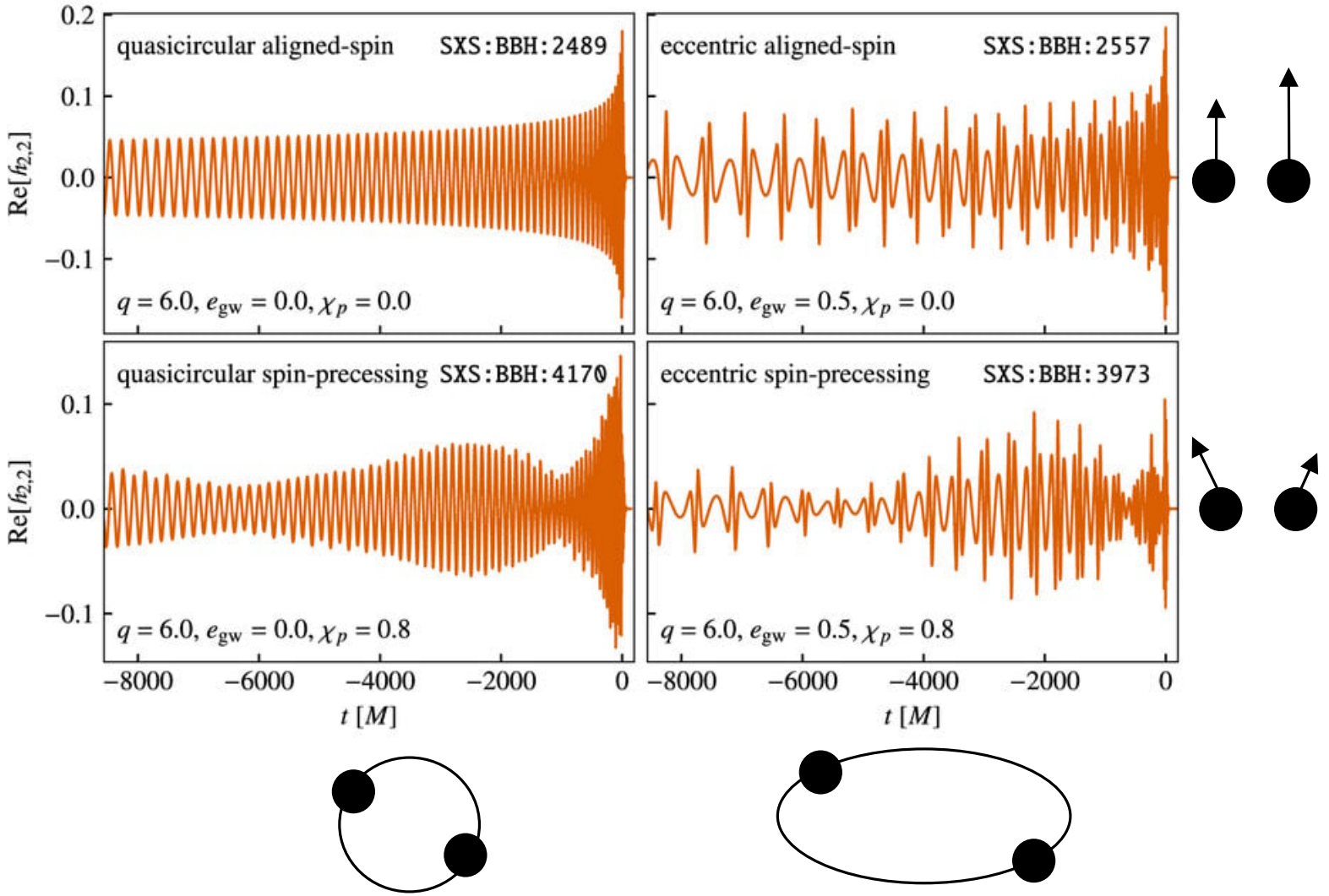
But this is very challenging!

No one technique can span the whole parameter space

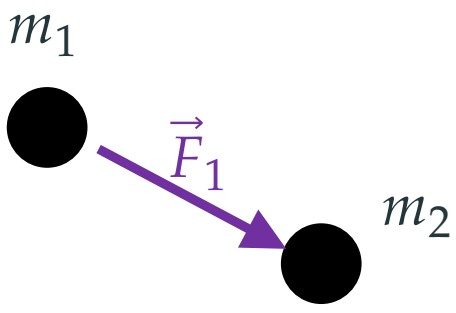
Parameter space



Parameter space

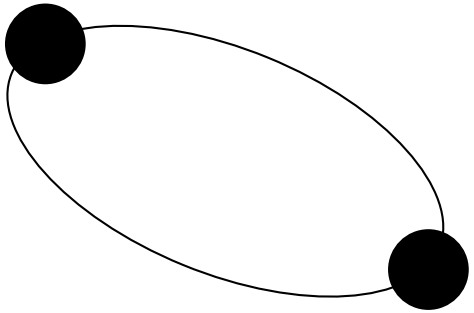


Post-Newtonian: 0PN order (Newtonian)



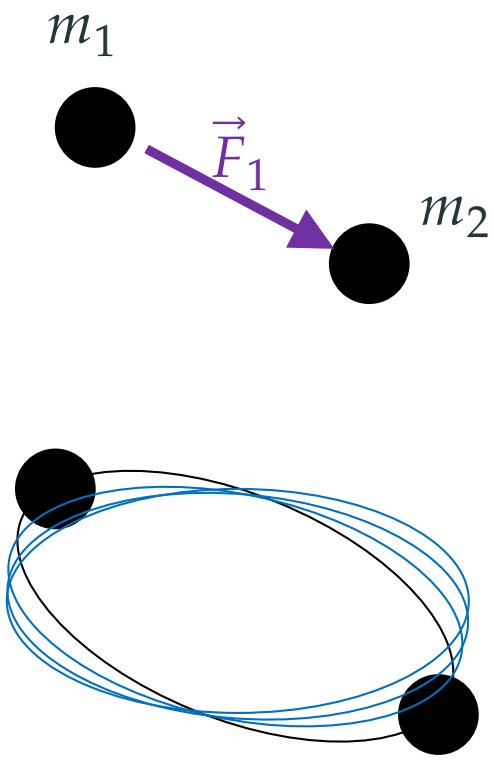
Newton's law of gravity

$$\overrightarrow{F_1} = m_1 \overrightarrow{a_1}$$
$$= m_1 \times \frac{G m_2}{r_{12}^2} \vec{n}_{12}$$



Solution for bound orbits:
ellipse (Kepler's laws)

Post-Newtonian: 1PN order (post-Newtonian)



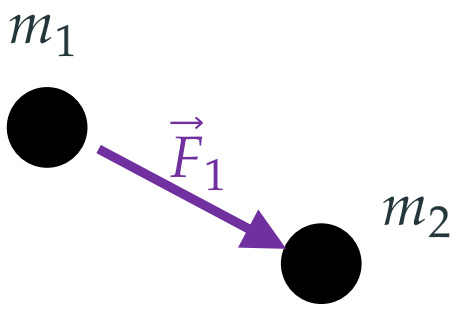
Einstein-Infeld-Hoffmann equations

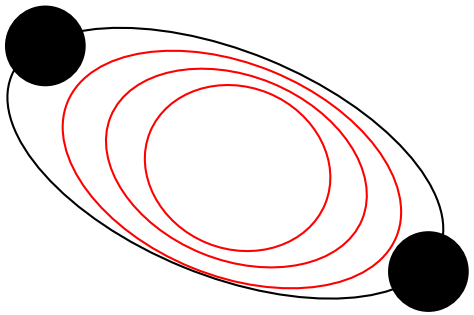
$$\overrightarrow{F_1} = m_1 \overrightarrow{a_1}$$
$$= m_1 \times \frac{G m_2}{r_{12}^2} \vec{n}_{12}$$
$$+ \frac{m_1}{c^2} \delta a_1^{EIH}(m_1, m_2, r_{12}, \overrightarrow{y_1}, \overrightarrow{y_2}, \overrightarrow{v_1}, \overrightarrow{v_2})$$

Solution for bound orbits:
periastron precession
(conservative effect)

7

Post-Newtonian: 2.5PN order (radiation reaction)





Radiation reaction (dissipative)

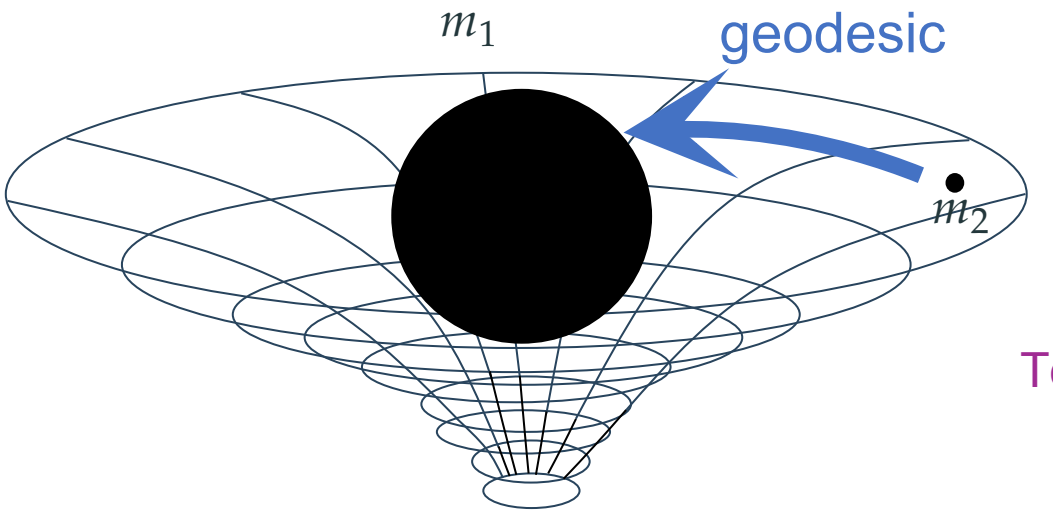
$$\overrightarrow{F_1} = m_1 \overrightarrow{a_1} \quad + \frac{1}{c^2}(1\text{PN}) + \frac{1}{c^4}(2\text{PN}) + \frac{1}{c^5}(2.5\text{PN})$$

Due to gravitational radiation

Solution for bound orbits:

- contraction of the orbit (*inspiral*)
- circularization of the orbit

Gravitational self-force: 0SF order (geodesic)

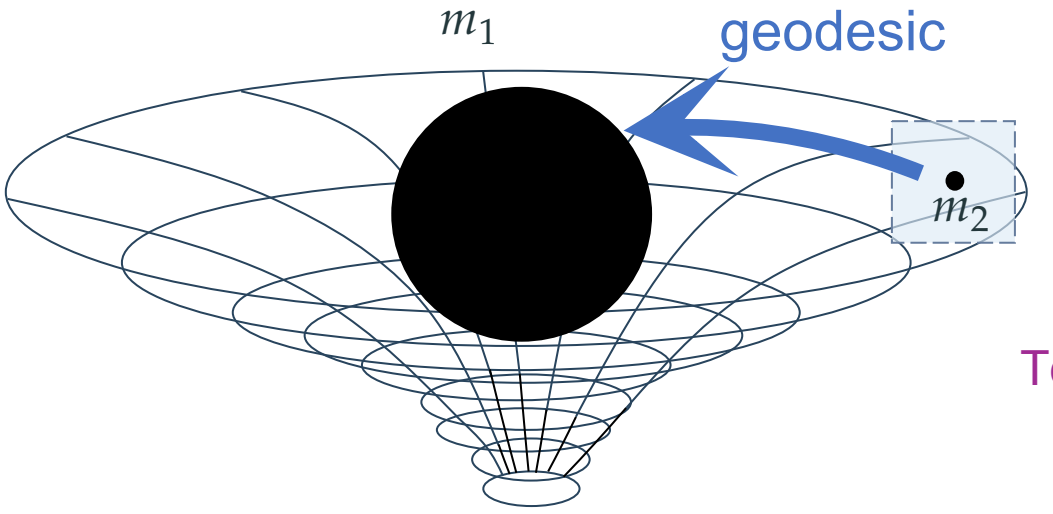


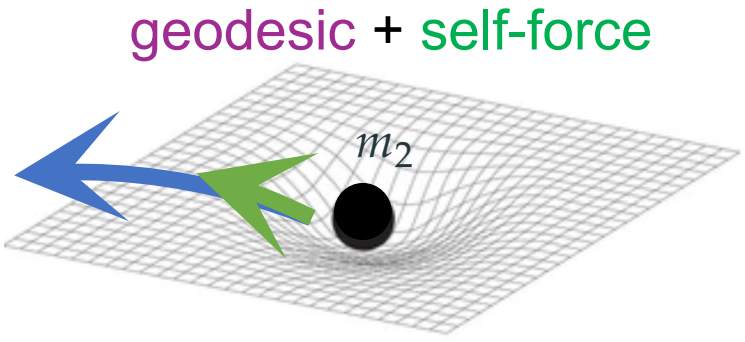
Test particle limit:

$$\frac{m_2}{m_1} \to 0$$

10

Gravitational self-force: 1SF order (geodesic)





Test particle limit:

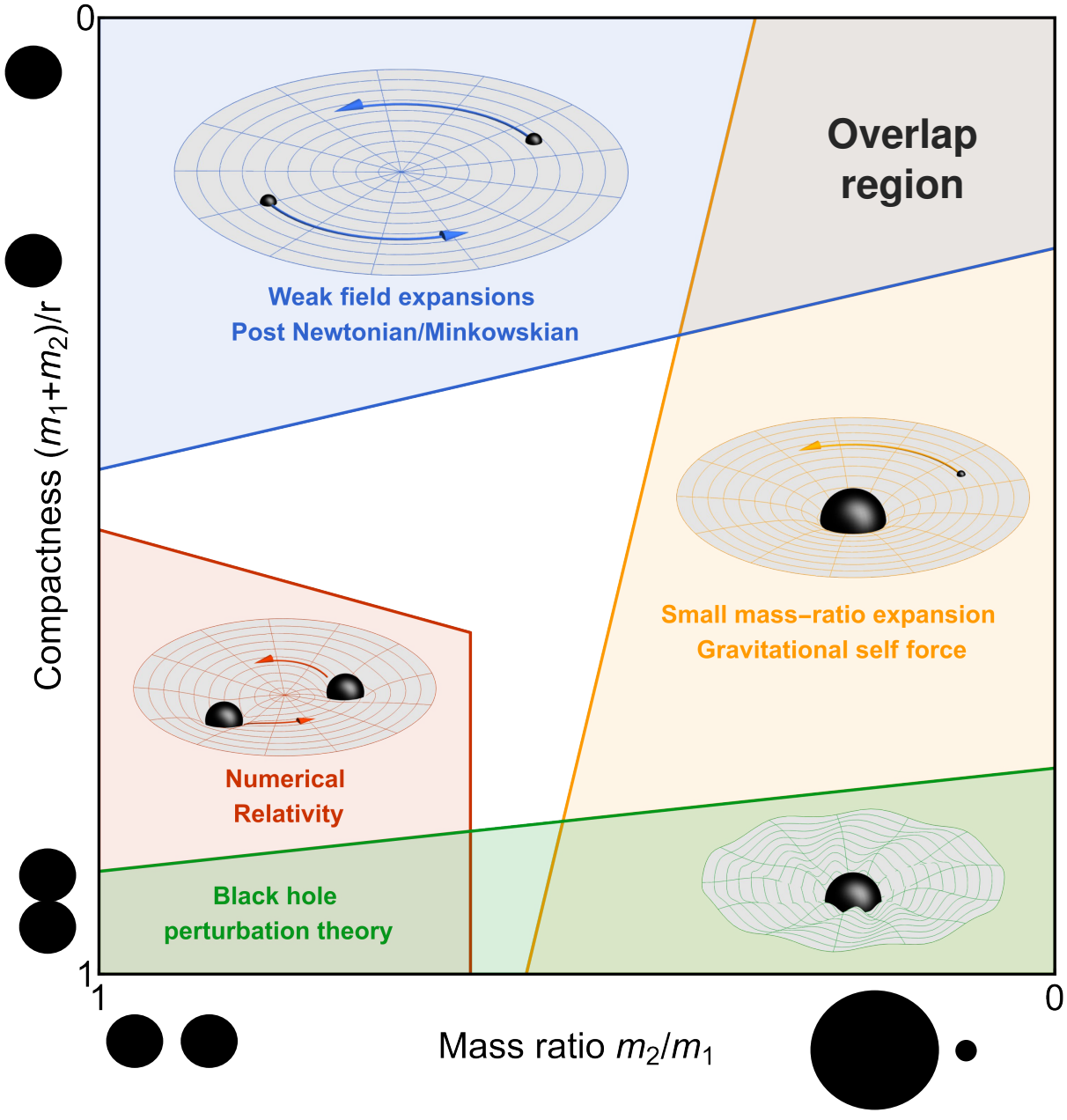
$$\frac{m_2}{m_1} \ll 1$$

+ correction due to its finite mass:

self-force $\propto \frac{m_2}{m_1}$

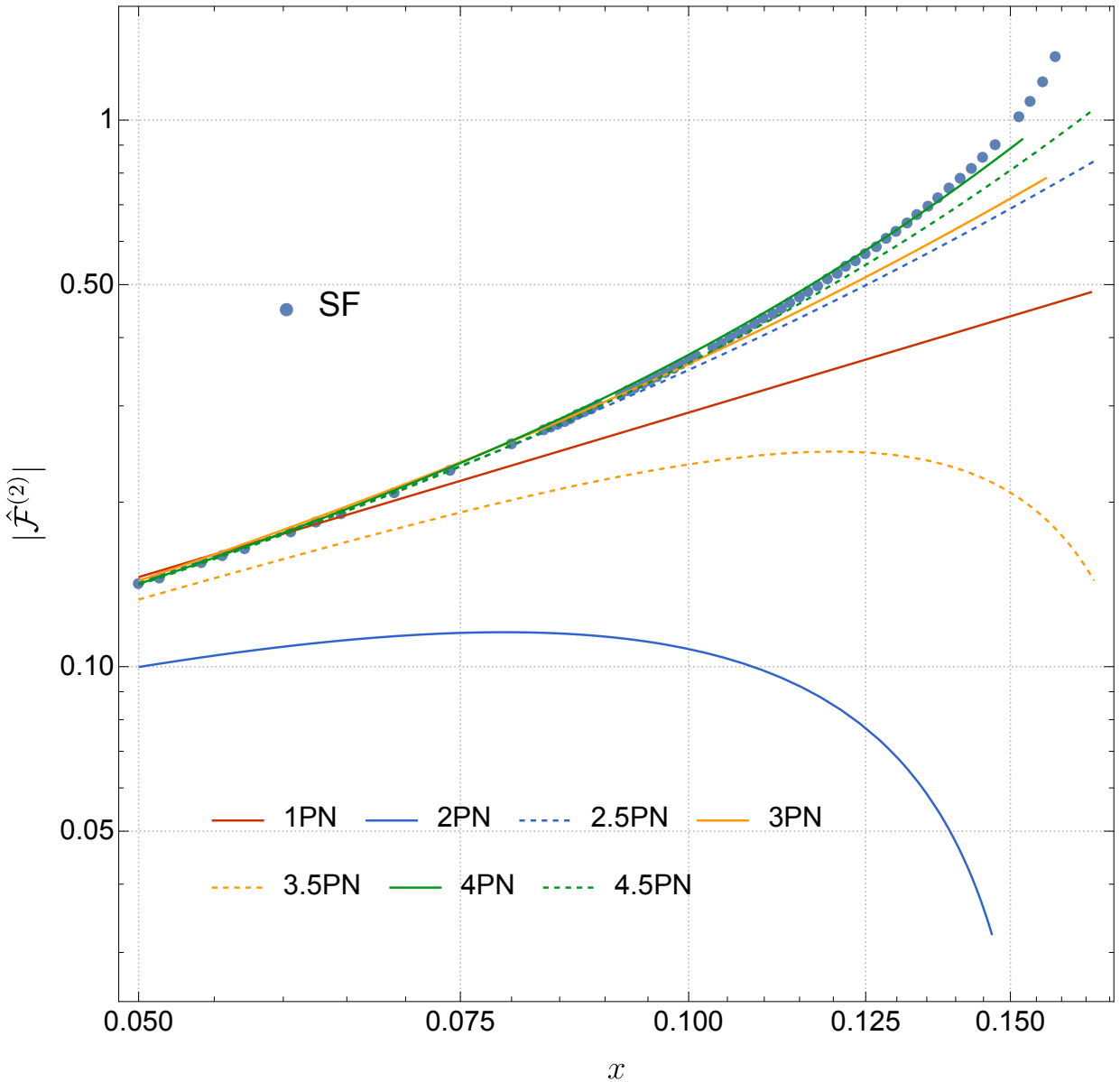
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Overlap region



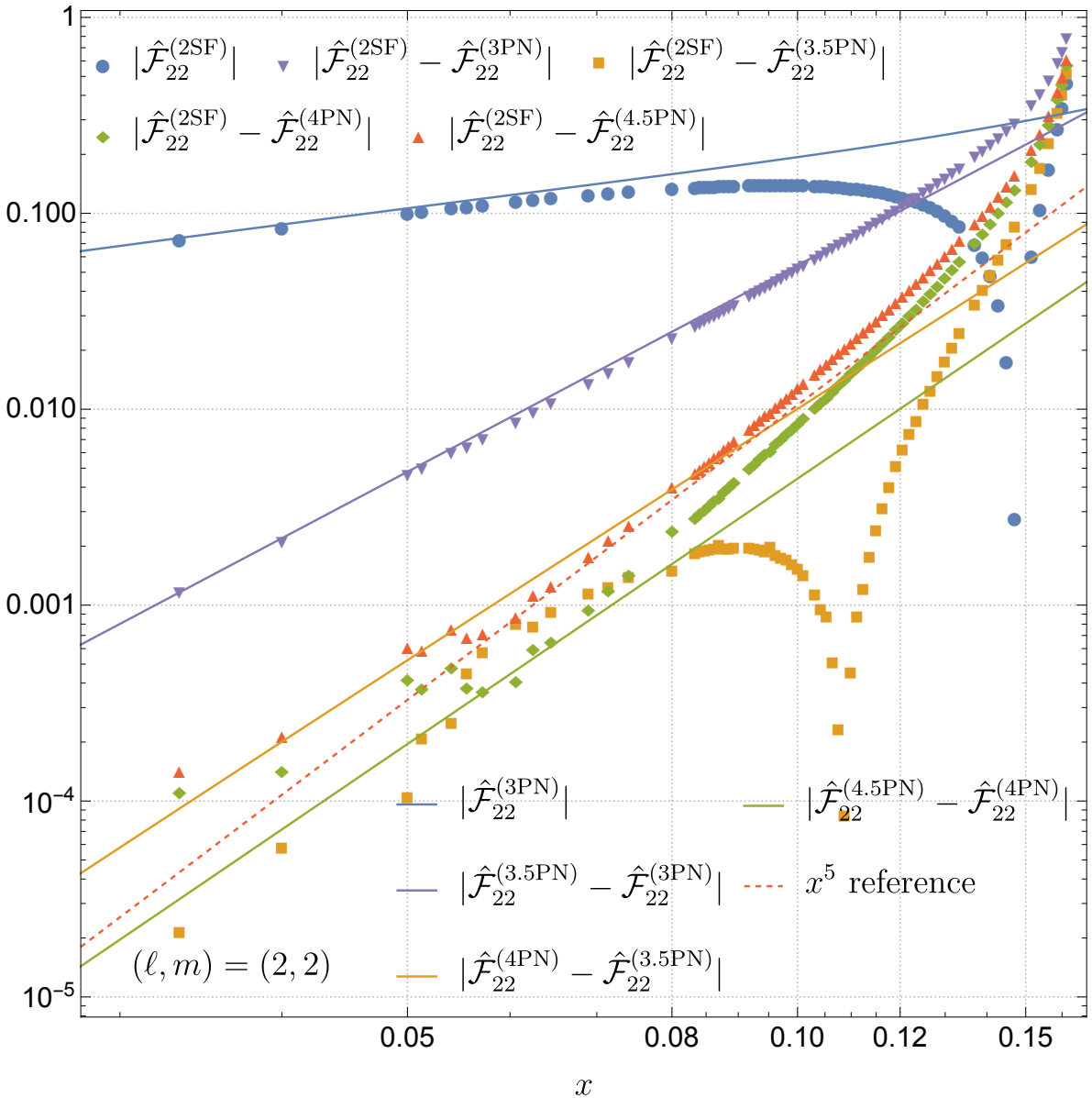
Comparison of the total flux (4PN vs 2SF)

From [Warburton, Wardell, DT, Henry, Pound, Blanchet et al. 2026]



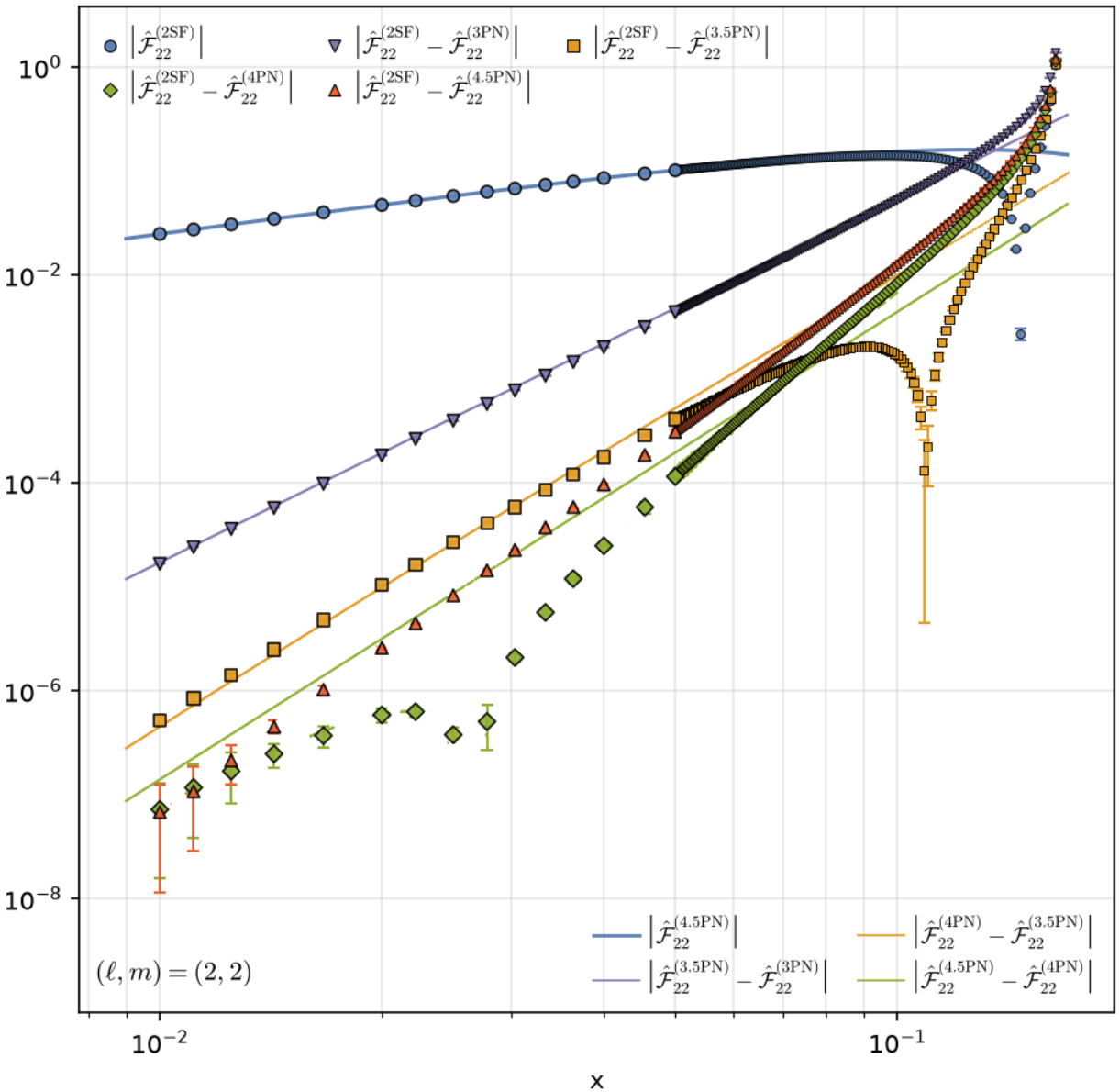
Comparison of the $(2,2)$ mode of the flux (4PN vs 2SF)

From [Warburton, Wardell, DT, Henry, Pound, Blanchet et al. 2026]

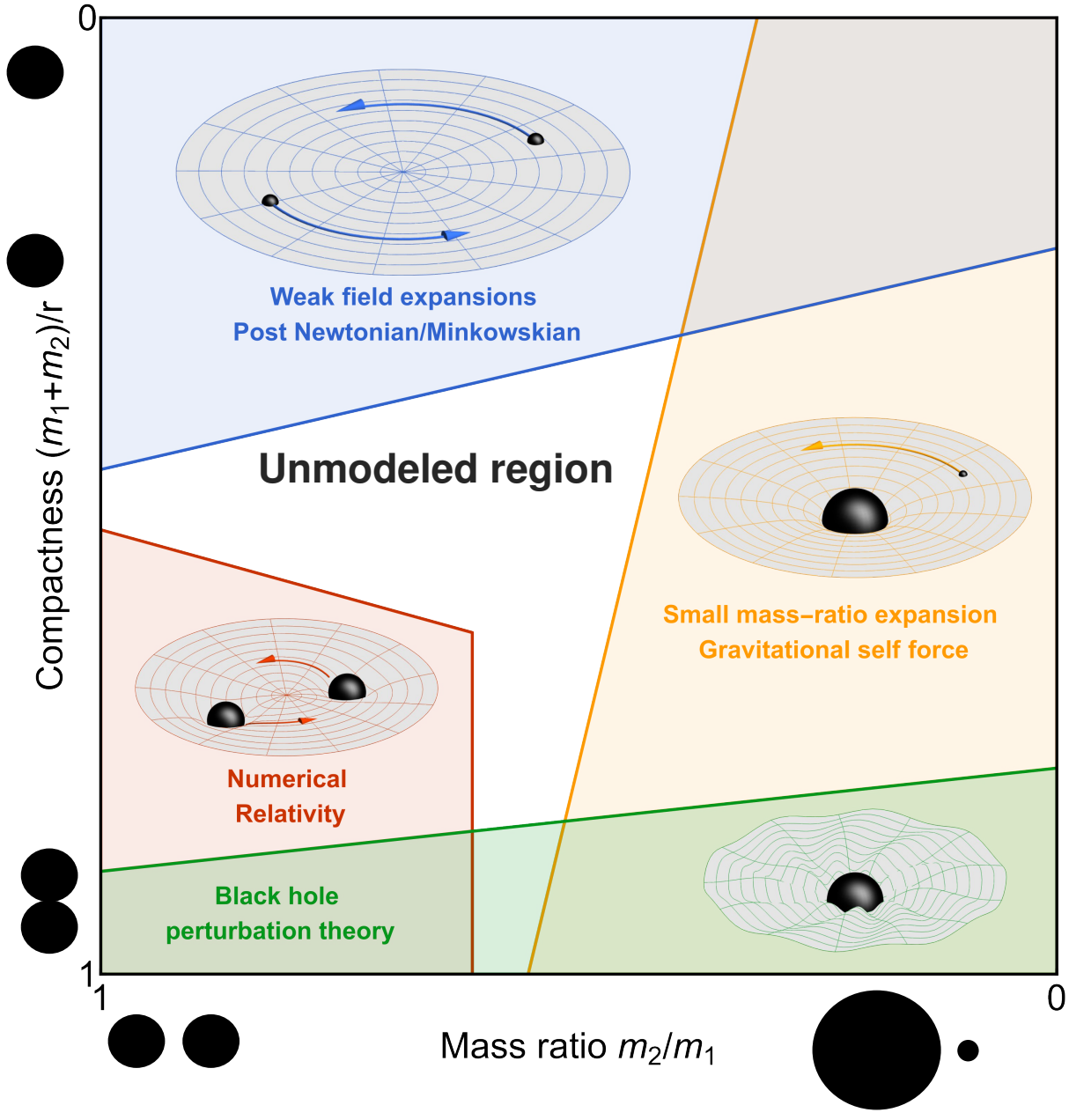


Improved comparison of the flux 4PN/2SF

Courtesy: Barry Wardell



Unmodelled region



Obtaining the phase for circular orbits

Consider circular orbits with orbital frequency Ω_ϕ

Two main quantities we want to compute in perturbation theory:

- binding/conservative energy $E(\Omega_\phi)$
- flux of energy radiated by the binary $\mathcal{F}(\Omega_\phi)$

One can then deduce the frequency evolution ("chirp") from the flux-balance law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\Omega_\phi}{\mathrm{d}t} = -\frac{\mathcal{F}(\Omega_\phi)}{\mathrm{d}E/\mathrm{d}\Omega_\phi}$$

This is an ODE on Ω_ϕ! Orbital phase given by $\phi = \int^t \mathrm{d}t'\, \Omega_\phi(t')$

The structure of the phasing for circular orbits

Define $x = \Omega_\phi^{2/3}$ and $\nu = m_1 m_2/(m_1+m_2)^2$

The number of cycles to merger has the structure

$$\begin{aligned}\mathcal{N} &= \frac{\phi_{\mathsf{ISCO}} - \phi}{2\pi} \\ &= \frac{1}{\nu}\left[A_{\mathsf{0PN}}\, x^{-5/2} + A_{\mathsf{1PN}}\, x^{-3/2} + A_{\mathsf{2PN}}\, x^{-1/2} + A_{\mathsf{2.5PN}}\, x^{0} + \cdots\right] \\ &\quad + B_{\mathsf{1PN}}\, x^{-3/2} + B_{\mathsf{2PN}}\, x^{-1/2} + B_{\mathsf{2.5PN}}\, x^{0} + \cdots \\ &\quad + \nu\left[C_{\mathsf{2PN}}\, x^{-1/2} + C_{\mathsf{2.5PN}}\, x^{0} + \cdots\right] + \nu^2\left[C_{\mathsf{3PN}}\, x^{+1/2} + \cdots\right]\end{aligned}$$

The structure of the phasing for circular orbits

Define PN parameter $x = \Omega_\phi^{2/3}$ and $\nu = m_1 m_2/(m_1+m_2)^2$

The number of cycles to merger has the structure

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When $\nu \ll 1$:

Small error on the 0PA coefficient (e.g. due to the PN approximation) causes a large dephasing

The structure of the phasing for circular orbits

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When $\nu \ll 1$:

Neglecting 1PA contribution leads to errors of $\mathcal{O}(1)$

Small error on the 1PA coefficient (e.g. due to the PN approximation) causes small dephasing

The structure of the phasing for circular orbits

Define PN parameter $x = \Omega_\phi^{2/3}$ and $\nu = m_1 m_2/(m_1+m_2)^2$

The number of cycles to merger has the structure

$$\begin{aligned}\mathcal{N} &= \frac{\phi_{\rm ISCO} - \phi}{2\pi} \\ &= \frac{1}{\nu}\Big[A_{\rm 0PN}\, x^{-5/2} + A_{\rm 1PN}\, x^{-3/2} + A_{\rm 2PN}\, x^{-1/2} + A_{\rm 2.5PN}\, x^{0} + \cdots\Big] \\ &\quad + B_{\rm 1PN}\, x^{-3/2} + B_{\rm 2PN}\, x^{-1/2} + B_{\rm 2.5PN}\, x^{0} + \cdots \\ &\quad + \nu\Big[C_{\rm 2PN}\, x^{-1/2} + C_{\rm 2.5PN}\, x^{0} + \cdots\Big] + \nu^2\Big[C_{\rm 3PN}\, x^{+1/2} + \cdots\Big]\end{aligned}$$

When $\nu \ll 1$:

For long inspiral ($x \ll 1$), neglecting the 2PA contribution leads to potentially important dephasing : $\mathcal{O}(\nu x^{-1/2})$

If approximated by PN, errors negligeable: $\mathcal{O}(\nu x^N)$ with $N \geq 0$.

The structure of the phasing for circular orbits

Define PN parameter $x = \Omega_\phi^{2/3}$ and $\nu = m_1 m_2/(m_1+m_2)^2$

The number of cycles to merger has the structure

$$\begin{aligned}\mathcal{N} &= \frac{\phi_{\text{ISCO}} - \phi}{2\pi} \\ &= \frac{1}{\nu}\Big[A_{\text{0PN}}\, x^{-5/2} + A_{\text{1PN}}\, x^{-3/2} + A_{\text{2PN}}\, x^{-1/2} + A_{\text{2.5PN}}\, x^{0} + \cdots\Big] \\ &\quad + B_{\text{1PN}}\, x^{-3/2} + B_{\text{2PN}}\, x^{-1/2} + B_{\text{2.5PN}}\, x^{0} + \cdots \\ &\quad + \nu\Big[C_{\text{2PN}}\, x^{-1/2} + C_{\text{2.5PN}}\, x^{0} + \cdots\Big] + \nu^2\Big[C_{\text{3PN}}\, x^{+1/2} + \cdots\Big]\end{aligned}$$

When $\nu \ll 1$:

3PA terms are always negligeable $\sim \mathcal{O}(\nu x^N)$ with $N \geq 0$.

1PA models for circular orbits

A purely self-force waveform is obtained as [0PA in green, 1PA in blue]

$$E(\Omega_\phi) \approx \nu \frac{1-2\Omega_\phi^{2/3}}{\sqrt{1-3\Omega_\phi^{2/3}}} + \nu^2 E_{(1)}^{\text{GSF num}}(\Omega_\phi)$$

$$\mathcal{F}(\Omega_\phi) \approx \nu^2 \mathcal{F}_{(0)}^{\text{GSF num}}(\Omega_\phi) + \nu^3 \mathcal{F}_{(1)}^{\text{GSF num}}(\Omega_\phi)$$

Note that $\mathcal{F}_{(1)}^{\text{GSF num}}(\Omega_\phi)$ was extremely challenging to obtain and is ony known for circular orbits in Schwarzschild [Wardell et al., 2112.12265]

Inject this into $\mathrm{d}\Omega_\phi/\mathrm{d}t = -\mathcal{F}(\Omega_\phi)/(\mathrm{d}E/\mathrm{d}\Omega_\phi)$: obtain "1PAT1" phasing

Computing $\mathcal{F}_{(1)}^{\text{GSF num}}$ for eccentric orbits will be even more challenging

2PA terms are uncontrolled in this approach

Hybrid models for circular orbits

Approximate the Ω_ϕ-depend coefficient in the ν-expansion by the best available approximation (exact, GSF, or PN) [0PA in green, 1PA in blue, higher PA in black].

$$E(\Omega_\phi) \approx \nu \frac{1-2\Omega_\phi^{2/3}}{\sqrt{1-3\Omega_\phi^{2/3}}} + \nu^2 E_{(1)}^{\text{GSF num}}(\Omega_\phi) + \sum_{k=2}^{k_{\max}} \nu^k E_{(k)}^{\text{PN}}(\Omega_\phi)$$

$$\mathcal{F}(\Omega_\phi) \approx \nu^2 \mathcal{F}_{(0)}^{\text{GSF num}}(\Omega_\phi) + \nu^3 \mathcal{F}_{(1)}^{\text{PN}}(\Omega_\phi) + \sum_{k=2}^{k_{\max}} \nu^k \mathcal{F}_{(k)}^{\text{PN}}(\Omega_\phi)$$

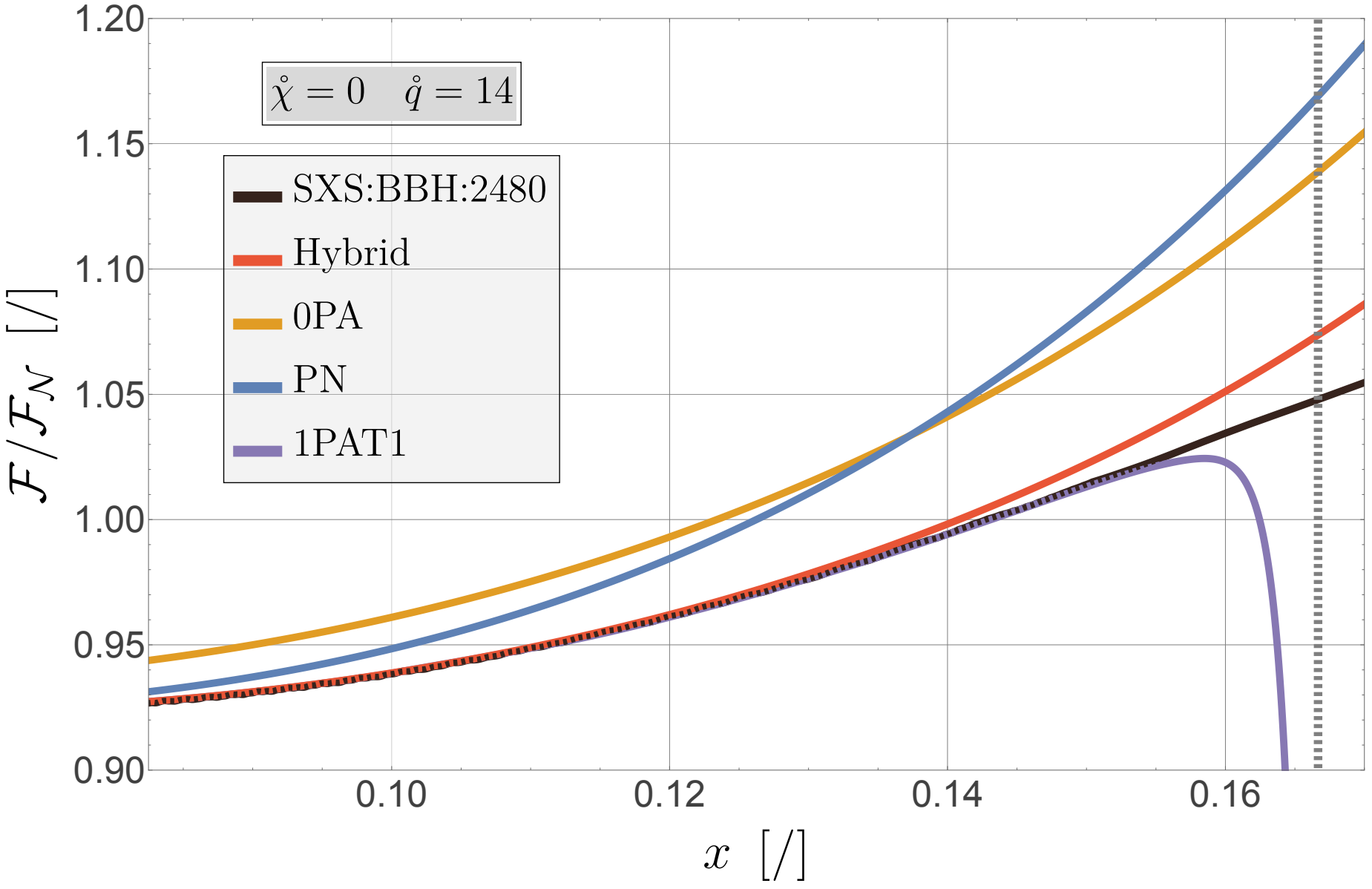
Weak field not covered by numerical grids: use high-order PN instead

Inject this into $\mathrm{d}\Omega_\phi/\mathrm{d}t = -\mathcal{F}(\Omega_\phi)/(\mathrm{d}E/\mathrm{d}\Omega_\phi)$ to obtain phasing

Very successful in the circular case [Honet, Pound & Compère, 2510.16114]

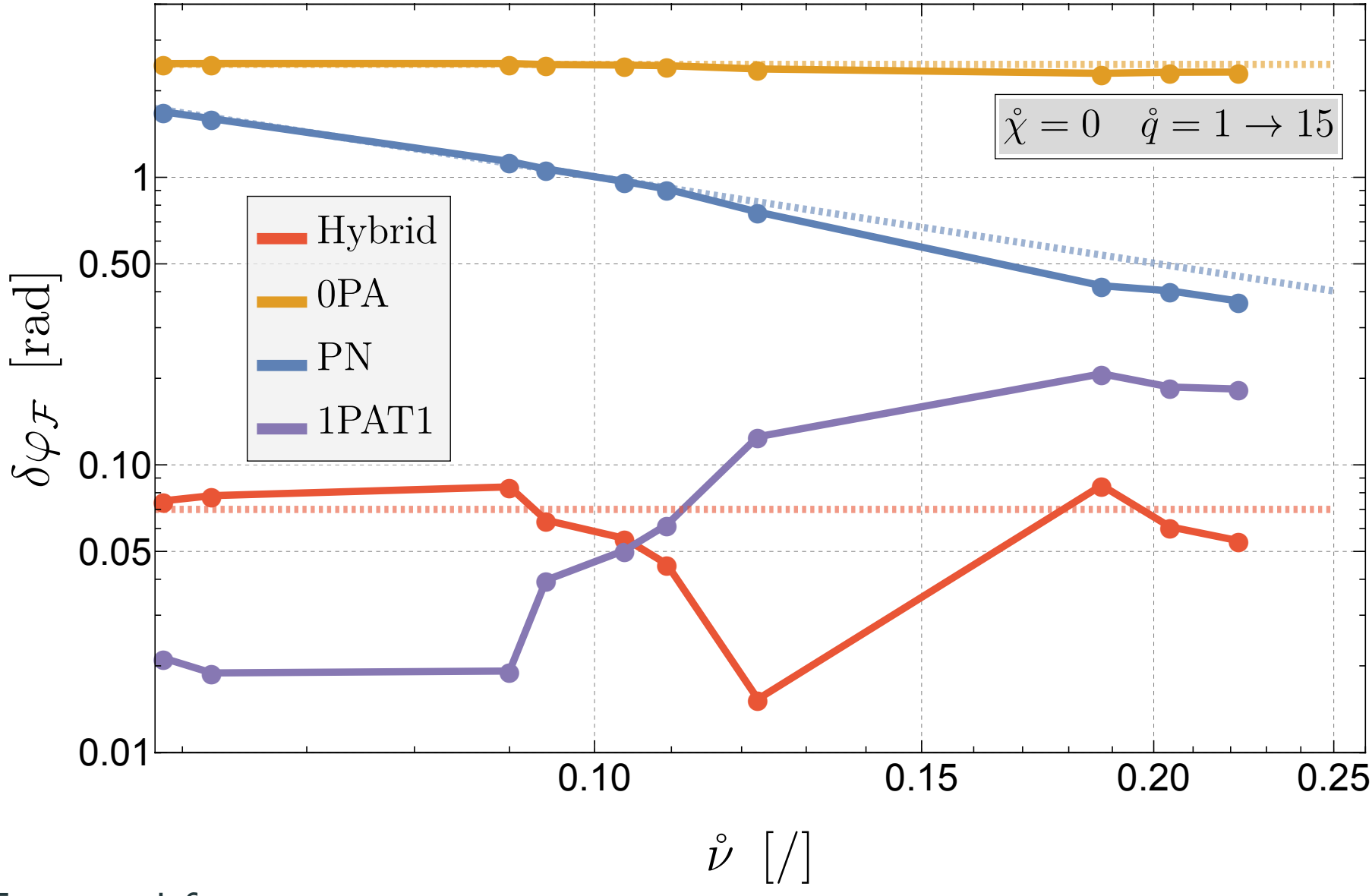
$\Rightarrow$ uses 4.5PN flux [Blanchet, Faye, Henry, Larrouturou & DT 2023] and 4PN binding energy [Damour, Jaranowski & Schäfer 2014][Bernard, Blanchet, Faye & Marchand 2018], including the 4PN Schott term [DT 2025]

Flux comparison: NR, PN, 0PA, 1PA and hybrid



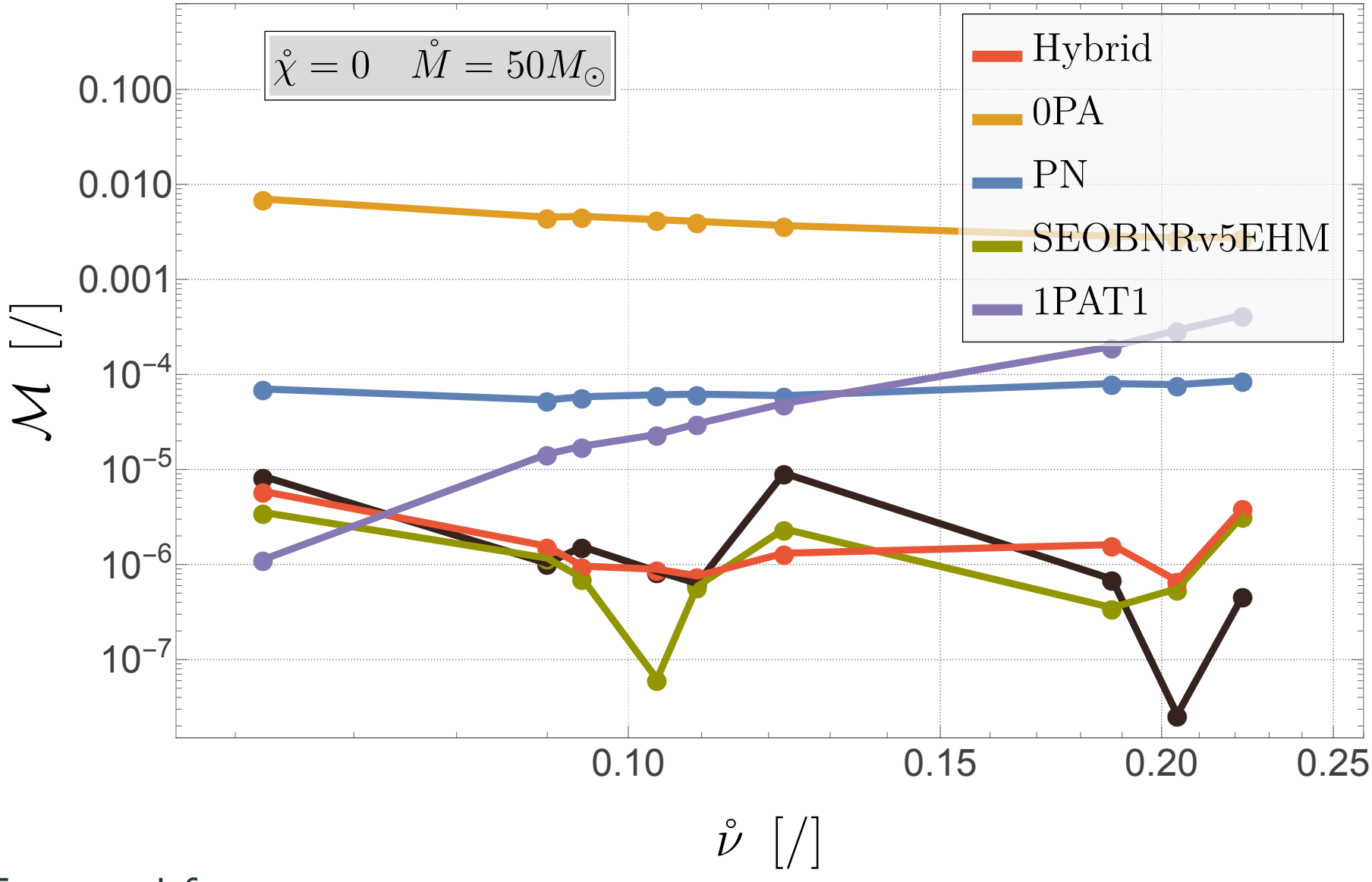
Extracted from [Honet, Pound & Compère 2510.16114]

Dephasing comparison: NR, PN, 0PA, 1PA and hybrid



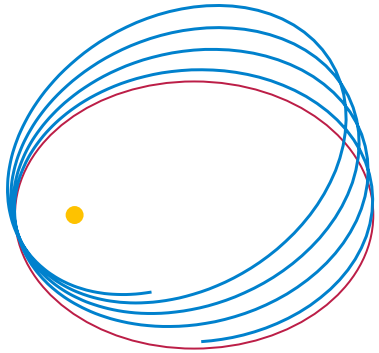
Extracted from [Honet, Pound & Compère 2510.16114]

Mismatch comparison: EOB, PN, 0PA, 1PA and hybrid



Extracted from [Honet, Pound & Compère 2510.16114]

Describing an eccentric orbit



An eccentric orbit (without spin) can be described by any two of the following gauge-invariant quantities:

- energy $E = H(p_r, p_\phi, r)$
- angular momentum $L = p_\phi$
- radial action $J_r = (2\pi)^{-1} \oint p_r \mathrm{d}r$
- azimuthal frequency $\Omega_\phi = 2\pi/T_\phi$ (where T_ϕ is the period for the separation vector to rotate by 2π)
- radial frequency $\Omega_r = 2\pi/T_r$ (where T_r is the period between to periastron passages)

Waveform generation for eccentric orbits

The phasing is obtained from the two flux-balance laws:

$$\frac{\mathrm{d}E(\Omega_\phi,\Omega_r)}{\mathrm{d}t} = -\mathcal{F}_E(\Omega_\phi,\Omega_r) \quad \text{and} \quad \frac{\mathrm{d}L(\Omega_\phi,\Omega_r)}{\mathrm{d}t} = -\mathcal{F}_L(\Omega_\phi,\Omega_r)$$

This is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Omega_\phi\\ \Omega_r\end{pmatrix} = -\begin{pmatrix}\frac{\partial E}{\partial \Omega_\phi} & \frac{\partial E}{\partial \Omega_r}\\ \frac{\partial L}{\partial \Omega_\phi} & \frac{\partial L}{\partial \Omega_r}\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{F}_E\\ \mathcal{F}_L\end{pmatrix}$$

These are two coupled ODEs on (Ω_ϕ,Ω_r)

Waveform generation for eccentric orbits

For a 1PA waveform, two ingredients are needed [0PA in green, 1PA in blue]:

- the constants of motion in terms of frequencies

$$E(\Omega_\phi, \Omega_r) = \nu E_{(0)}(\Omega_\phi, \Omega_r) + \nu^2 E_{(1)}(\Omega_\phi, \Omega_r)$$
$$L(\Omega_\phi, \Omega_r) = \nu L_{(0)}(\Omega_\phi, \Omega_r) + \nu^2 L_{(1)}(\Omega_\phi, \Omega_r)$$

  I obtained these at 4PN in [DT 2025] and at 1SF (numerical and 9PN) in [DT, Nasipak & Pound 2025]

- the fluxes in terms of frequencies

$$\mathcal{F}_E(\Omega_\phi, \Omega_r) = \nu^2 E_{(0)}(\Omega_\phi, \Omega_r) + \nu^3 \mathcal{F}_E(1)(\Omega_\phi, \Omega_r)$$
$$\mathcal{F}_L(\Omega_\phi, \Omega_r) = \nu^2 \mathcal{F}_L(0)(\Omega_\phi, \Omega_r) + \nu^3 \mathcal{F}_L(1)(\Omega_\phi, \Omega_r)$$

  These are known to 3PN [Arun, Blanchet, Iyer & Sinha 2009]

Issue #1 in parametrization: evolve (Ω_ϕ, Ω_r) or (E, L)?

If we want to implement directly

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Omega_\phi \\ \Omega_r\end{pmatrix} = -\begin{pmatrix}\frac{\partial E}{\partial \Omega_\phi} & \frac{\partial E}{\partial \Omega_r} \\ \frac{\partial L}{\partial \Omega_\phi} & \frac{\partial L}{\partial \Omega_r}\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{F}_E \\ \mathcal{F}_L\end{pmatrix}$$

we run in to the following problem: even if we have reasonable numerical errors of the fluxes at energy corrections, the Jacobian can become extremely large and blow these errors up.

It is therefore preferable to directly evolve

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}E \\ L\end{pmatrix} = -\begin{pmatrix}\mathcal{F}_E(E, L) \\ \mathcal{F}_L(E, L)\end{pmatrix}$$

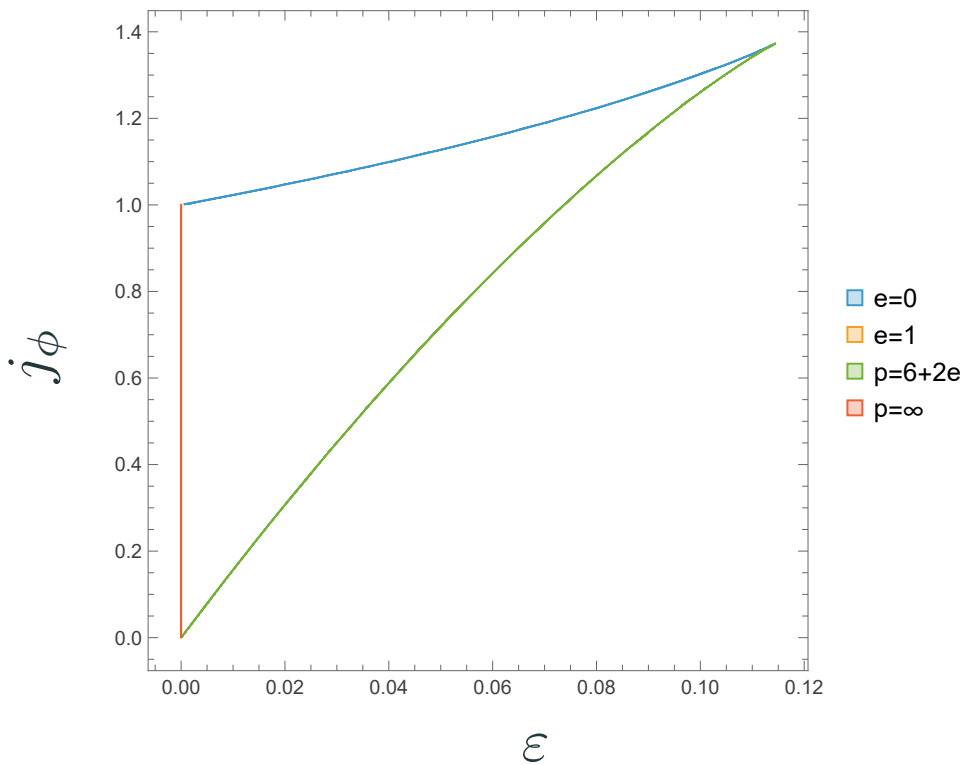
making it a coupling ODE on E and L. We then use the inverse map between (E, L) and (Ω_ϕ, Ω_r) to evolve the frequencies and obtain the phase.

Issue #2 in parametrization: circular orbit limit

We want to compare our eccentric model with the circular model in the limit of circular orbits. But how do we define this?

In the (E, L) plane, circular orbits are given by a $L_{\rm circ}(E)$ curve which is only known approximately (1SF or 4PN order).

Restricting to the $\nu \to 0$ limit, the allowed parameter space is:



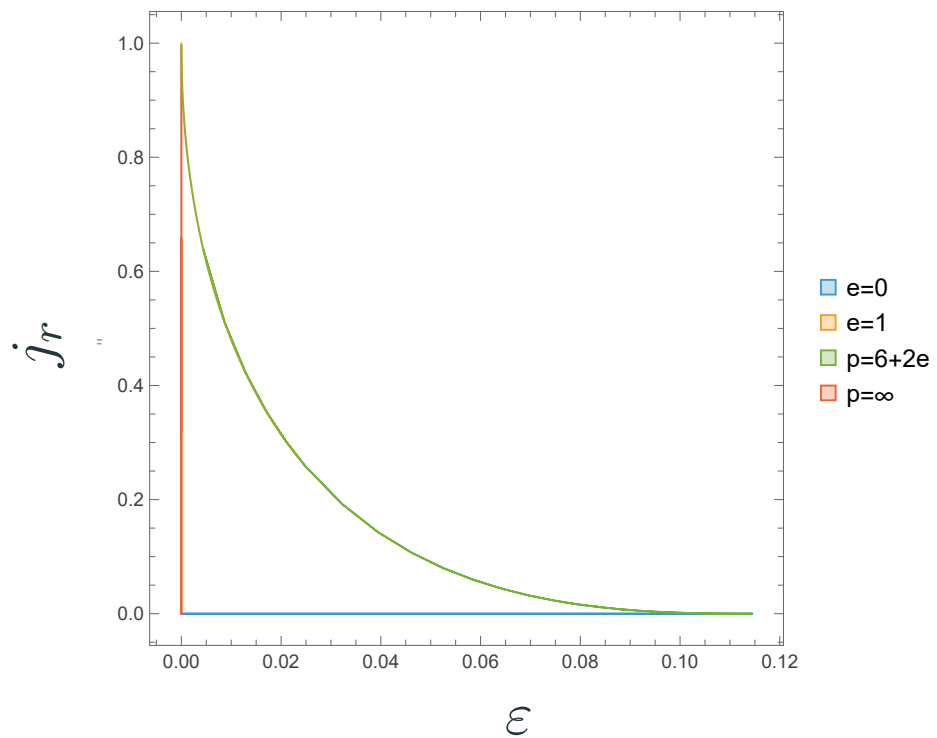
We define $\varepsilon = -2E/(m\nu)$ and $j_\phi = -2EL^2/(m^5\nu^3)$. Circular orbits correspond to the blue curve: one typically has $j_\phi > 1$. The GW amplitudes, when PN expanded in small ε, have $\sqrt{1-j}$ contributions, which are ill defined.

Solution: parametrize by radial action

To solve this problem, we propose to instead describe our motion with the energy E and the radial action J_r:

- E and J_r are gauge-invariant constants of motion
- by definition, the PN limit is $E \to 0$, the circular limit is $J_r \to 0$

Introducing the parameters $\varepsilon = -2E/(m\nu)$ and $j_r = -2EJ_r^2/(m^5\nu^3)$:



Flux-balance law for radial action

The time evolution of the radial action is obtained through

$$\frac{\mathrm{d}J_r}{\mathrm{d}t} = \frac{\partial J_r}{\partial E} \times \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial J_r}{\partial J} \times \frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{1}{\Omega_r} \times \mathcal{F} + \frac{\Omega_\phi}{\Omega_r} \times \mathcal{G}$$

This leads us to define

$$\mathcal{K} = \frac{\mathcal{F} - \Omega_\phi \, \mathcal{G}}{\Omega_r J_r}$$

such that

$$\frac{\mathrm{d}J_r}{\mathrm{d}t} = -J_r \, \mathcal{K}$$

The numerator and denominator in $\mathcal{K}$ both vanish in the circular limit.

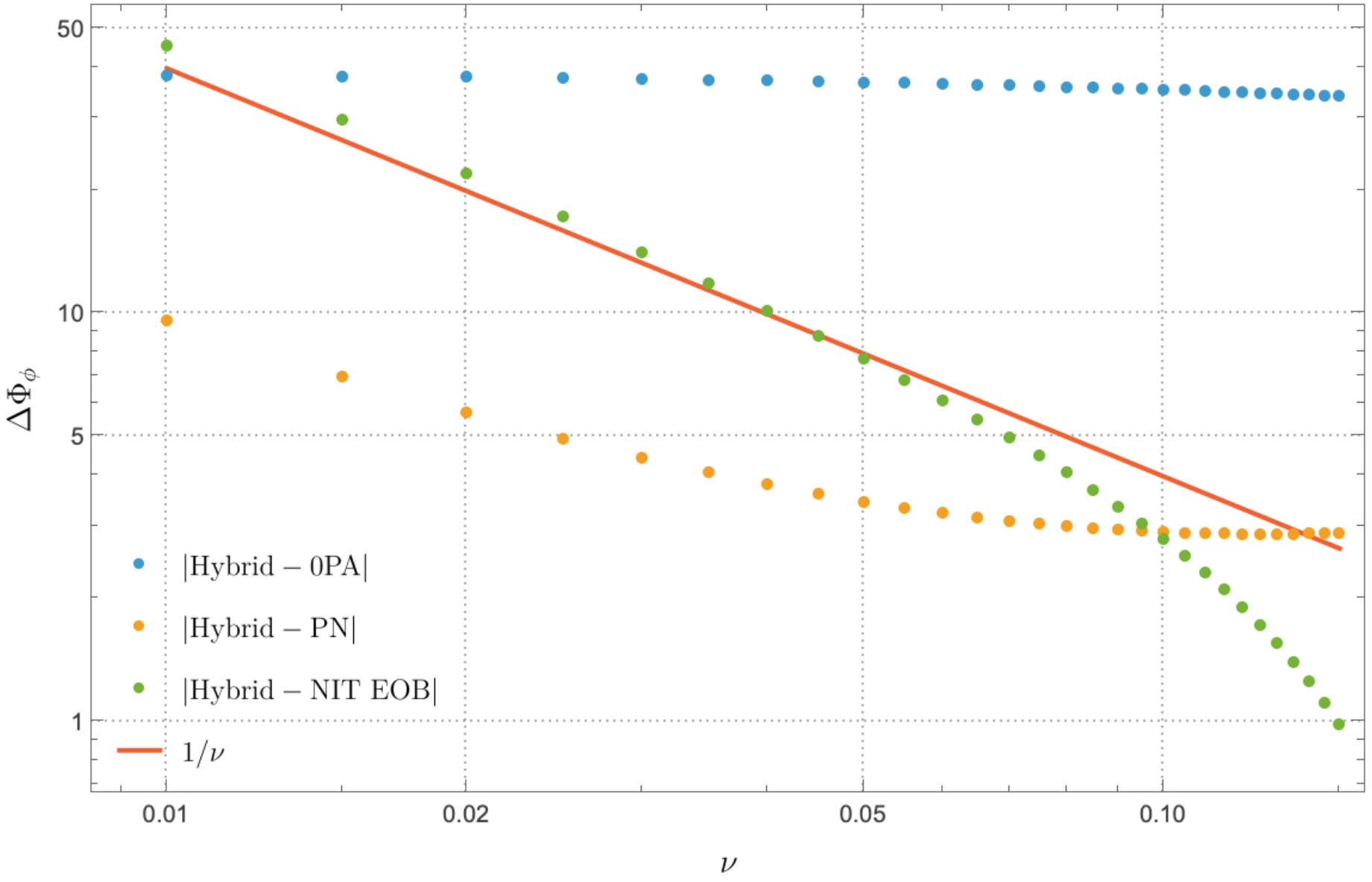
The ratio is finite and positive in the circular limit.

Challenge: J_r is well defined from the dynamics as an action variable, but is not a known asymptotic BMS quantity. It is unclear how to contruct Schott terms and justify rigorously the flux balance law.

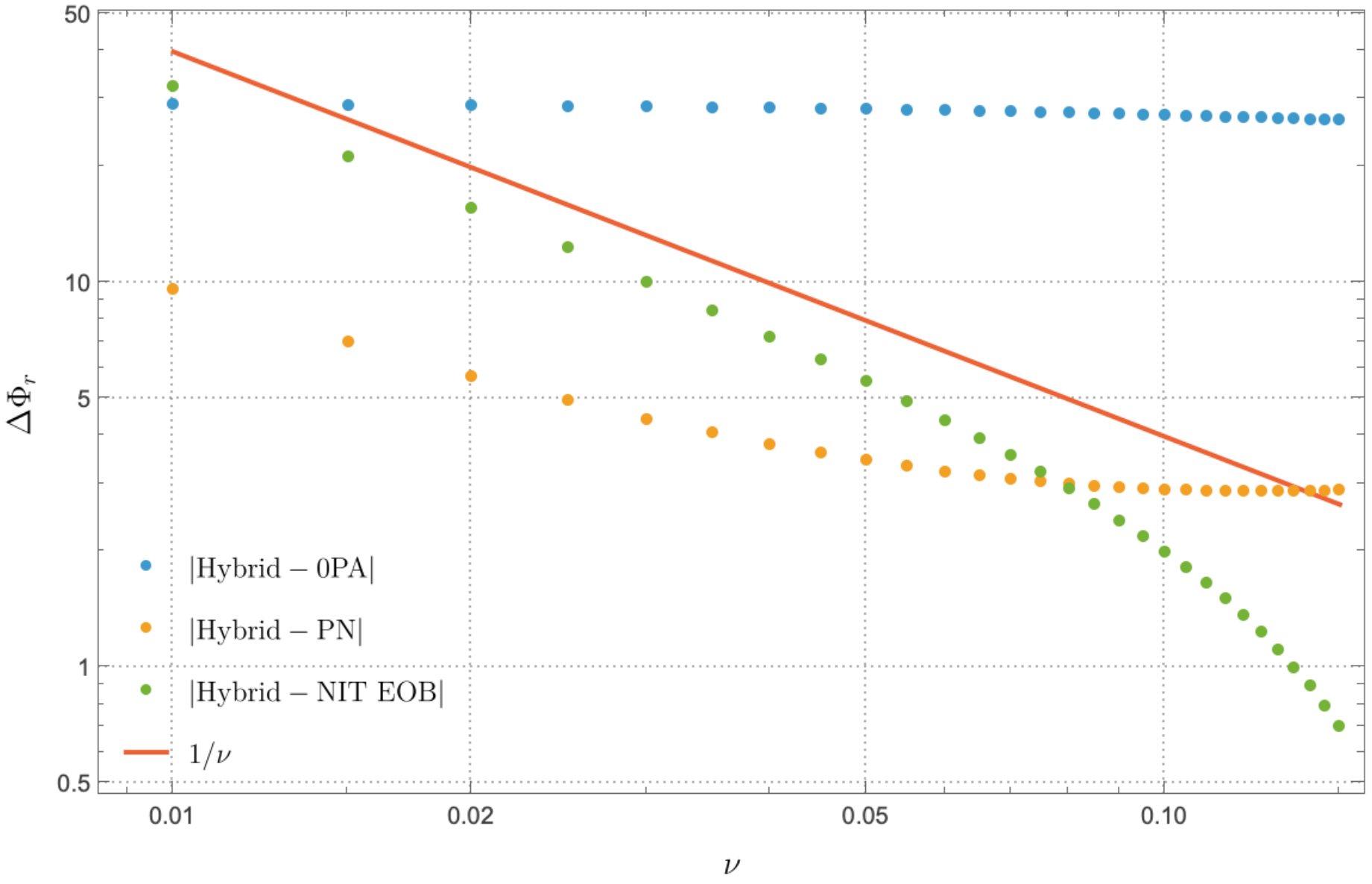
$\mathcal{K}$ for circular orbits

$$\begin{aligned}\mathcal{K}_{\text{circ}}(\varepsilon) = \frac{704}{15}\frac{\varepsilon^4\nu}{m}\Bigg\{1 &+ \varepsilon\left(-\frac{67}{308} - \frac{75}{22}\nu\right) + \frac{1081}{176}\pi\varepsilon^{3/2} \\ &+ \varepsilon^2\left(\frac{399211}{66528} - \frac{27245}{3696}\nu + \frac{383}{88}\nu^2\right) + \varepsilon^{5/2}\pi\left(\frac{8589}{1408} - \frac{391057}{9856}\nu\right) \\ &+ \varepsilon^3\Bigg[\frac{35842751009}{128066400} + \frac{817}{66}\pi^2 + \left(-\frac{682301}{1386} + \frac{22263}{1408}\pi^2\right)\nu \\ &\qquad + \frac{63027}{1232}\nu^2 - \frac{61}{22}\nu^3 + \frac{87419}{2310}\ln 2 - \frac{234009}{6160}\ln 3 - \frac{87419}{4620}\ln\left(16e^{2\gamma_{\mathsf{E}}}\varepsilon\right)\Bigg] \\ &+ \mathcal{O}(\varepsilon^4)\Bigg\}\end{aligned}$$

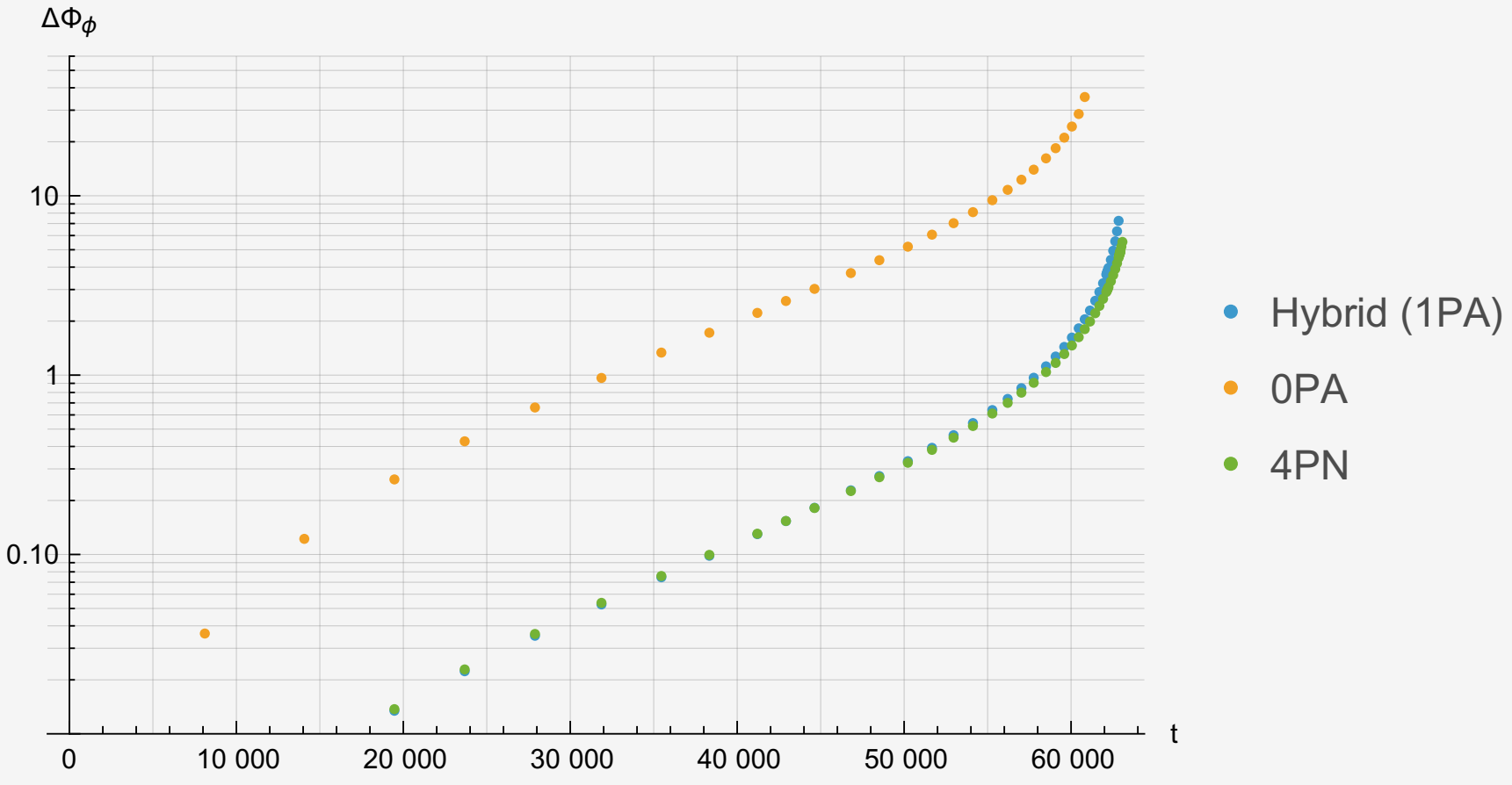
Dephasing $\Delta\phi_\phi$ for $(e = 0.4,\ p = 30 \to p = 15)$



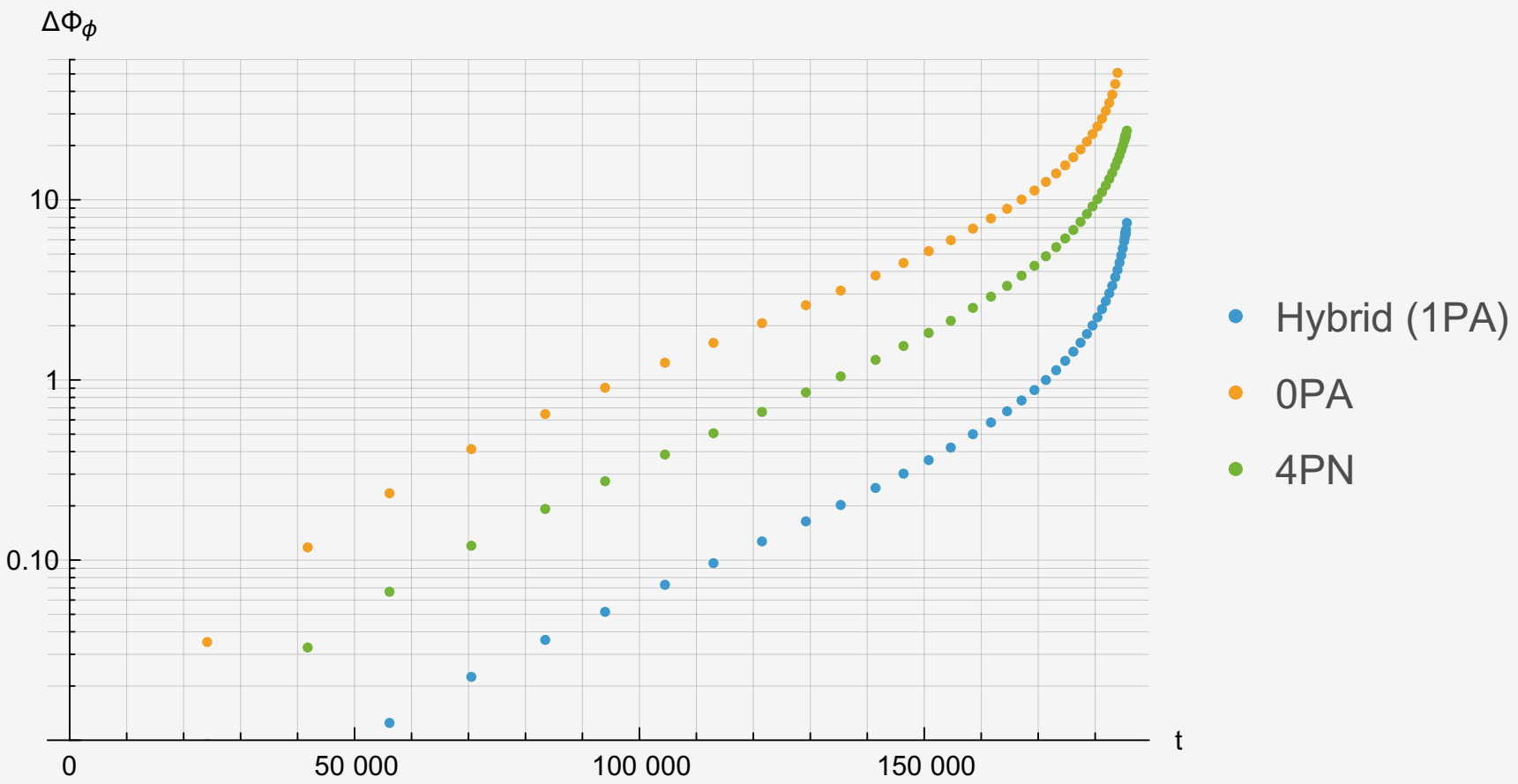
Dephasing $\Delta\phi_r$ ($e = 0.4$, $p = 30 \to p = 15$)



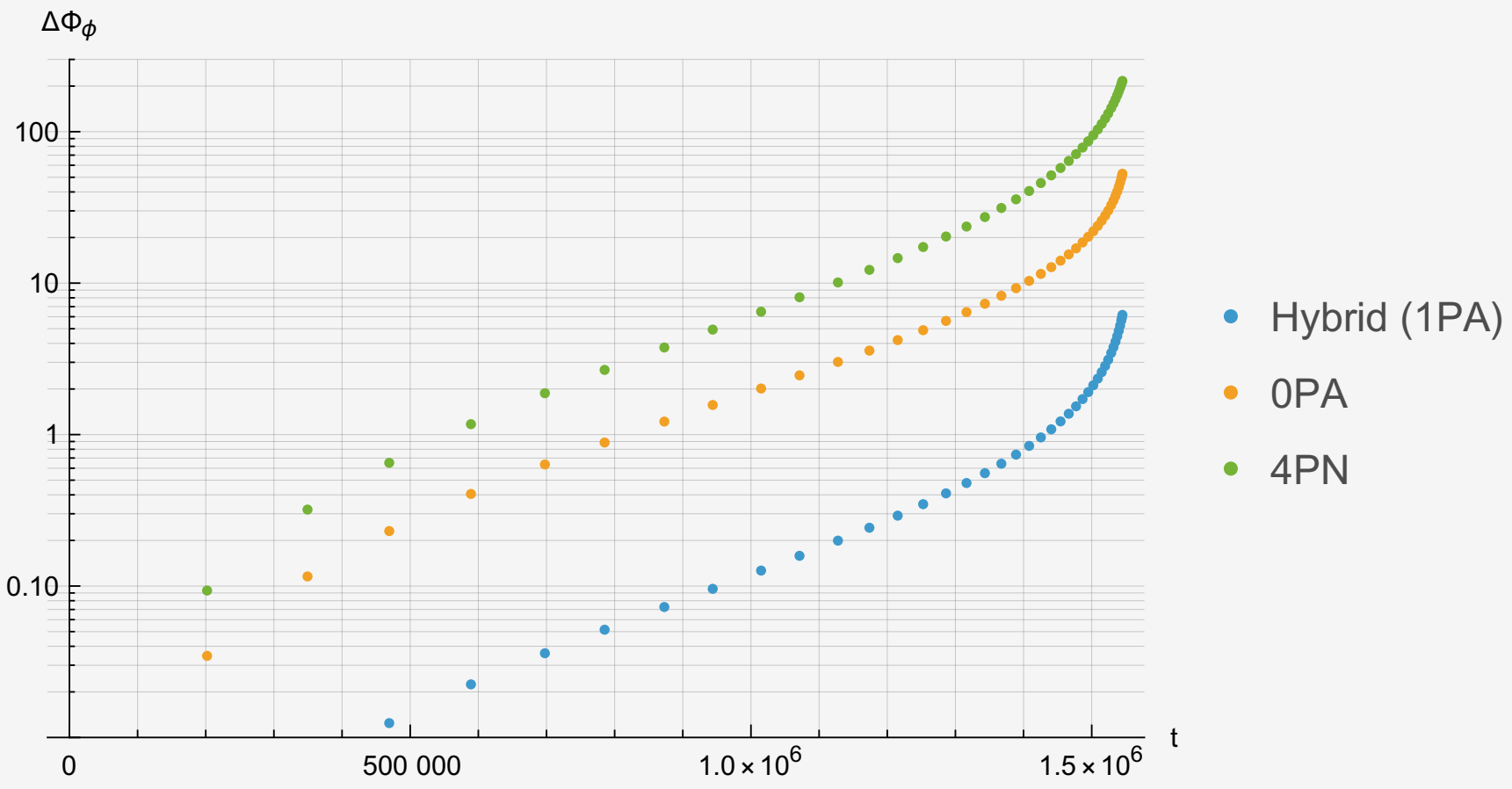
Dephasing $\Delta\phi_\phi$ wrt 1PAT1 ($e = 0$, $p = 30 \to p = p_{\text{min}}$) for $q = 1$



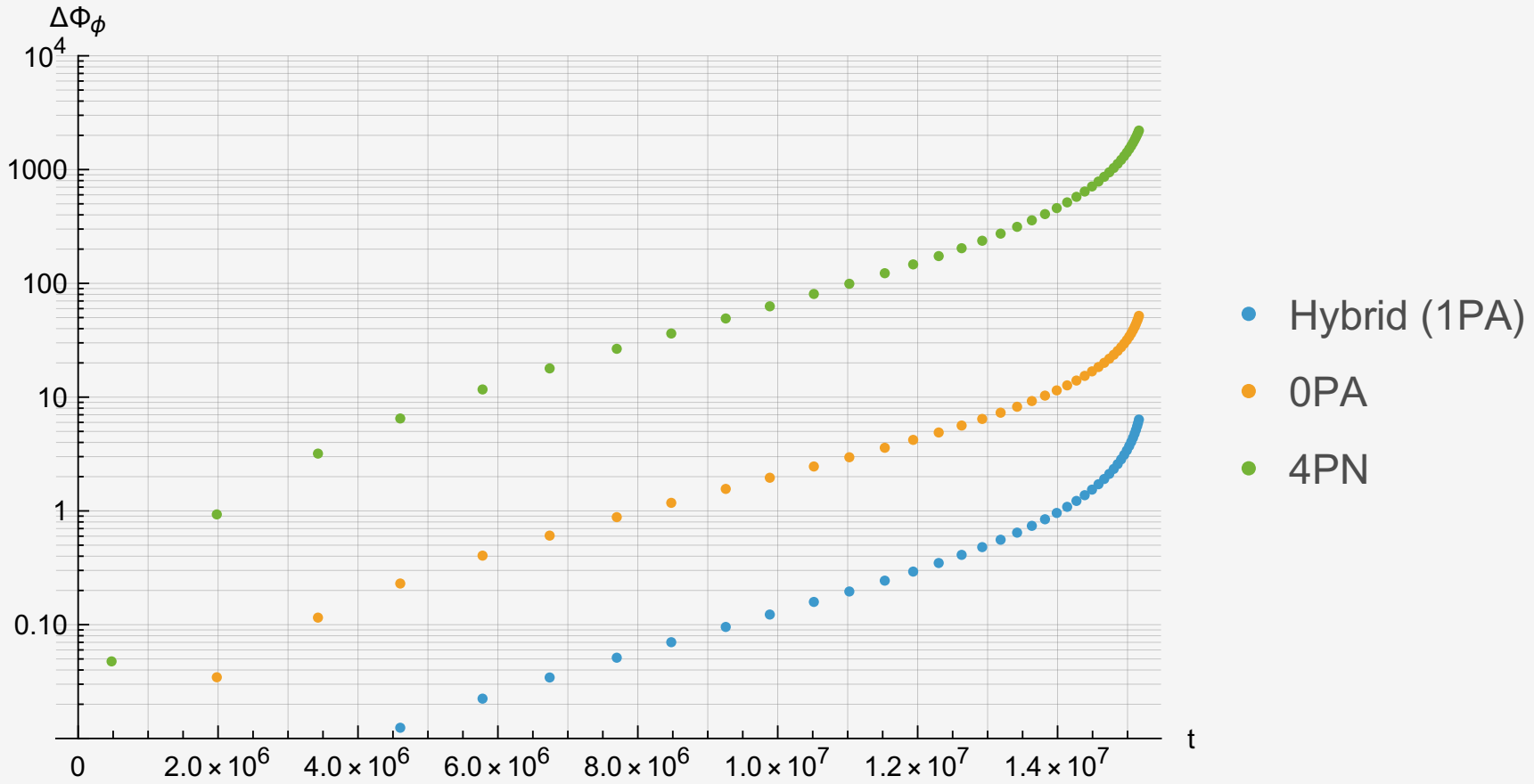
Dephasing $\Delta\phi_\phi$ wrt 1PAT1 ($e = 0$, $p = 30 \to p = p_{\text{min}}$) for $q = 10$



Dephasing $\Delta\phi_\phi$ wrt 1PAT1 ($e = 0$, $p = 30 \to p = p_{\text{min}}$) for $q = 100$

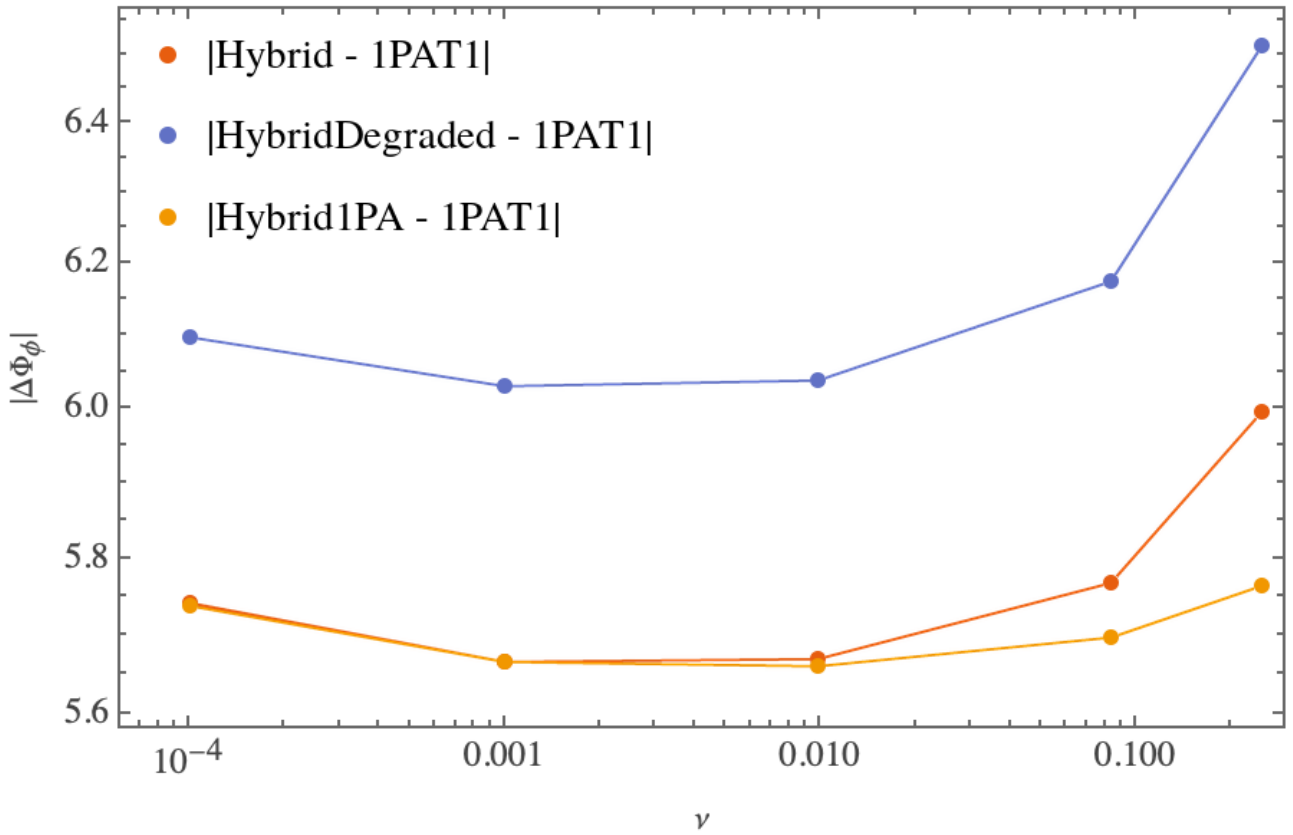


Dephasing $\Delta\phi_\phi$ wrt 1PAT1 ($e = 0$, $p = 30 \to p = p_{\text{min}}$) for $q = 1000$



Dephasing $\Delta\phi_\phi$ for circular orbits ($p = 30 \to p = 10$)

Hybrid: model described. **1PA:** remove $\geq$ 2PA contributions from PN.
Degraded: replace numerical 1SF constants-to-frequency map by PN.



Consistent with a 1PA error due to neglected higher PN terms, but dephasing is large compared to the circular model.

Ideas to solve this problem

We do not understand why the error is consistent but much larger than the circular model. Things we want to test:

- check if including the 4PN flux terms in the circular limit helps (answer: only very slightly)
- check if including the 1PA numerical flux leads to negligible errors
- check if a circular model based on evolving E rather than Ω_ϕ also has large errors
- attempt a (Ω_ϕ, J_r) parametrization to mimic the circular model (but do we get large Jacobians/large cancellations?)

Conclusion

- Post-Newtonian and self-force are consistent and complimentary expansions
- Hybridizing the two can yield results that are superior to either of these two methods
- This program is particularly important for intermediate mass ratio modelling (IMRIs) but also for EMRIs
- Challenges remain in how to perform this hybridization: many arbitrary choices which can strongly affect accuracy!
- Expanding the parameter space to precessing orbits requires better PN results than currently available