Non-relativistic CFTs and gravity

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Motivation

Holographic models for strongly coupled systems

- The AdS/CFT correspondence allows us to probe the physics of strongly coupled gauge theories.
  - Insight into transport properties of QGP, relevant for physics seen in heavy-ion collisions.
- There are other strongly coupled systems discussed in condensed matter literature which exhibit a wide range of extremely interesting physics.
- Use holographic methods to find the classical “Master field” for these theories.
Motivation

New insights into Quantum Gravity

- AdS/CFT has a dual role: it allows us to probe quantum aspects of gravity in terms of a non-perturbatively well defined QFT.
- Generalizations of the AdS/CFT correspondence, to new terrains has the potential to unveil important lessons for quantum gravity.

Understanding fluid dynamics

- The mathematical structure of Navier-Stokes equations (non-relativistic) poses interesting challenges.
- Can we reformulate the Fluid-Gravity correspondence in a context relevant for non-relativistic fluids?

Bhattacharyya, Hubeny, Minwalla, MR
Introduction

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Motivation

Experimental relevance

- There is currently an intensive experimental effort to understand the physics of cold atoms.
- These systems seem to admit an hydrodynamic description in terms of a nearly-ideal fluid.
  - The energy per particle is about 50% of the free value, similar in spirit to the Stephan-Boltzmann saturation of QGP just above the deconfinement transition.
  - Experimental results of elliptic type flow (shear driven relaxation) give $\eta/s \sim 1/\pi$!
- Can we find systems that have holographic duals which share at least some of the symmetries exhibited in these cold atom systems?
1 Introduction

2 Galilean Conformal Symmetry

3 A Holographic construction

4 Black holes and thermodynamics

5 Hydrodynamics

6 Discussion
References

- Proposal for holographic duals
  - Son: 0804.3972
  - Balasubramanian K, McGreevy: 0804.4053
- Holographic embedding in string theory, etc..
  - Herzog, MR, Ross: 0807.1099
  - Maldacena, Martelli, Tachikawa: 0807.1100
  - Adams, Balasubramanian K, McGreevy: 0807.1111
- Fluid dynamics
  - MR, Ross, Son, Thompson: 0711.2049
- Related work
  - Goldberger: 0705.2867
  - Barbon, Fuertes: 0705.3244
- Earlier relevant work
  - Nishida, Son: 0706.3746
  - Hubeny, MR, Ross: hep-th/0504034
Light-cone reductions

• Recall that one can get the Galilean algebra in d dimensions by reducing the Poincaré algebra $\text{SO}(d + 1, 1)$ on light-cone

$$u = t + y \ , \quad v = t - y$$

• Propagation in light-cone time $u$ respects Galilean invariance.

• We can similarly reduce the conformal algebra $\text{SO}(d + 2, 2)$ in $d + 2$ dimensions on a light-cone to obtain the Schrödinger algebra in $d$-spatial dimensions.
Starting from the conformal algebra we keep all generators which commute with the particle number.

- Hamiltonian: $H$
- Spatial rotations: $M_{ij}$
- Spatial momenta: $P_i$
- Galilean boosts: $K_i$
- Dilatation: $D$
- Special conformal generator: $C$
- Particle number: $N$

where we are restricting attention to $d$-spatial dimensions, i.e., $\{i, j\} \in \{1, \cdots d\}$.  

The Schrödinger algebra: Generators
### The Schrödinger algebra from conformal algebra

<table>
<thead>
<tr>
<th>Generator</th>
<th>Galilean</th>
<th>Conformal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particle number</td>
<td>$N$</td>
<td>$P_v$</td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>$H$</td>
<td>$P_u$</td>
</tr>
<tr>
<td>Momenta</td>
<td>$P_i$</td>
<td>$P_i$</td>
</tr>
<tr>
<td>Angular momenta</td>
<td>$M_{ij}$</td>
<td>$M_{ij}$</td>
</tr>
<tr>
<td>Galilean boost</td>
<td>$K_i$</td>
<td>$M_{iv}$</td>
</tr>
<tr>
<td>Dilatation</td>
<td>$D$</td>
<td>$D + M_{uv}$</td>
</tr>
<tr>
<td>Special conformal</td>
<td>$C$</td>
<td>$K_v$</td>
</tr>
</tbody>
</table>
Scaling dimensions and representations

- From the commutation relations descending from the conformal algebra one can infer that

\[
[H, D] = -2iH
\]

- This implies that the Hamiltonian has scaling dimension 2.
- Intuitively, this follows from the fact that non-relativistic systems are first order in time, leading to scaling

\[
t \rightarrow \lambda^2 t, \quad x \rightarrow \lambda x
\]
Scaling dimensions and representations

Aside: Lifshitz points

- We can also consider more general scaling, but not conformal symmetries.
- These are described by a real number $\nu$.
- The commutation relations are deformed to

$$ [D, H] = i (1 + \nu) H, \quad [D, N] = -i (\nu - 1) N $$

- For $\nu \neq 1$ we don’t have a conserved particle number and the special conformal generator $C$ does not exist in the algebra.
- These describe generalized scaling

$$ t \rightarrow \lambda^{1+\nu} t \quad x \rightarrow \lambda x $$
Consider the scaling symmetry

\[ t \rightarrow \lambda^{\nu+1} t , \quad x \rightarrow \lambda x \]

This can be achieved by starting from \( \text{AdS}_{d+3} \) in light-cone coordinates

\[ ds^2 = r^2 (-2\, du\, dv + dx^2) + \frac{dr^2}{r^2} \]

and define an unconventional scaling

\[ u \rightarrow \lambda^{\nu+1} u , \quad v \rightarrow \lambda^{1-\nu} v , \quad x \rightarrow \lambda x , \quad r \rightarrow \frac{1}{\lambda} r \]

and interpreting \( u \) as time.
This Galilean symmetry is familiar from DLCQ.

In fact, this is essentially the observation that DLCQ of any relativistic theory gives a Galilean invariant model in a sector with fixed light-cone momentum.

However, we should be careful about the zero mode.

Finally, the underlying theory is relativistic – the Galilean symmetry is an artifact of our choice of light-cone quantization.
Holography for non-relativistic CFTs

- To motivate a dual that has manifest Galilean scaling consider Son; Balasubramanian K, McGreevy

\[ ds^2 = r^2 \left( -2 \, du \, dv - \beta^2 \, r^2 \nu \, du^2 + dx^2 \right) + \frac{dr^2}{r^2} \]

which naturally has the required scaling

\[ u \rightarrow \lambda^{\nu+1} u , \quad v \rightarrow \lambda^{1-\nu} v , \quad x \rightarrow \lambda x , \quad r \rightarrow \frac{1}{\lambda} r \]

\begin{itemize}
  \item \( \nu = 0 \) is pure AdS\(_{d+3}\).
  \item \( \nu = 1 \) corresponds to the Schrödinger algebra.
  \item \( \nu = 2 \) is relevant for lightlike non-commutative SYM.
  \item We will call such spacetimes Schr\(_{d+3}\).
\end{itemize}
The metric with $\beta \neq 0$ is sourced by null energy momentum $T_{uu}$.

This can be shown to be a solution of Einstein-Hilbert action with negative cosmological constant, with a massive vector field providing the appropriate stress tensor.

In fact, this spacetime has naturally a Galilean causal structure.

Technically, it belongs to a class of spacetimes that is known as non-distinguishing.
Why is the spacetime non-distinguishing?

- The causal future of $p = (u_0, v_0, r_0, \vec{x}_0)$ is the set of points with $u > u_0$.
- So every point on a plane of constant $u$ shares the same causal future.
Why is the spacetime non-distinguishing?

- The geometry despite having local Lorentzian tangent space, achieves a global Galilean light-cone by its non-distinguishing character.
Realization in string theory

- The spacetime dual to Galilean CFTs can be generated from known solutions by a solution generating technique.
- This technique Null Melvin Twist or TsT transformation maps an asymptotically AdS geometry and converts it into a deformed spacetime with $\beta \neq 0$.

$$\text{TsT} = \text{T-duality} + \text{shift} + \text{T-duality}$$

- Starting from $\text{AdS}_{d+3} \times X$ with $X$ having one $U(1)$ isometry we generate $\text{Schr}_{d+3} \times_w X$. 
Realization in string theory

- Starting from AdS$_5 \times S^5$ and writing $S^5$ as $S^1$ fibration over CP$^2$ (with fibre $\psi$) we obtain via NMT

\[ ds^2 = r^2 \left( -2 \, du \, dv - r^2 \, du^2 + dx^2 \right) + \frac{dr^2}{r^2} + (d\psi + A)^2 + d\Sigma_4^2, \]

\[ F_{(5)} = 2 \left( 1 + \star \right) d\psi \wedge J \wedge J, \]

\[ B_{(2)} = r^2 \, du \wedge (d\psi + A), \]

- This geometry can be reduced to a solution of a 5 dimensional effective theory which is a consistent truncation of IIB supergravity involving a massive vector and 3 scalars. Maldacena, Martelli, Tachikawa
The Dual Field Theory

- The NMT also allows us to infer the dual field theory since we can follow the solution generating technique on the open string side.
- The field theory (for $\nu = 1$) is $N = 4$ SYM deformed by a (heterotic) star product

$$f \star g = e^{i\beta (\mathcal{V}^f R^g - \mathcal{V}^g R^f)} f g$$

where $\mathcal{V}$ is the $v$-momentum of the field and $R$ refers to a global $U(1)_R$ charge.
Effective Lagrangian

- For purposes of discussing thermodynamics issues we can however truncate to a one scalar model with action

\[
16\pi G_5 \mathcal{S} = \int d^5x \sqrt{-g} \left( R - \frac{4}{3} (\partial_\mu \phi) (\partial^\mu \phi) - V(\phi) \right) + \int d^5x \sqrt{-g} \left( \frac{1}{4} e^{-8\phi/3} F_{\mu\nu} F^{\mu\nu} - 4 A_\mu A^\mu \right)
\]

\[
V(\phi) = 4 e^{2\phi/3} (e^{2\phi} - 4)
\]

- This action needs to be supplemented with appropriate boundary terms.
Black Hole solution

\[ ds^2_E = r^2 k(r)^{-\frac{2}{3}} \left[ \left( \frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right) du^2 + \frac{\beta^2 r^4}{r^4} dv^2 - \left[ 1 + f(r) \right] du dv \right] \]

\[ + k(r)^{\frac{1}{3}} \left( r^2 dx^2 + \frac{dr^2}{r^2 f(r)} \right) , \]

\[ A = \frac{r^2}{k(r)} \left( \frac{1 + f(r)}{2} du - \frac{\beta^2 r^4}{r^4} dv \right) , \]

\[ e^\phi = \frac{1}{\sqrt{k(r)}} , \]

\[ f(r) = 1 - \frac{r^4_+}{r^4} , \quad k(r) = 1 + \frac{\beta^2 r^4_+}{r^2} \]
Thermodynamics

- The NMT/TsT does not change the entropy

\[ S = \frac{r_+^3 \beta}{4G_5} \Delta v V \]

- Note that the canonically normalized Killing generator of the horizon is

\[ \xi^a = \left( \frac{\partial}{\partial u} \right)^a + \frac{1}{2\beta^2} \left( \frac{\partial}{\partial v} \right)^a \]

- This gives the temperature:

\[ T = \frac{r_+}{\pi \beta} \]

- Moreover, the system is in a grand canonical ensemble with (particle number) chemical potential

\[ \mu = \frac{1}{2\beta^2} \]
Thermodynamics contd.

• To determine the Gibbs potential of this grand canonical ensemble, we can do an “Euclidean action” computation.

• Analytically continuation of $t$ gives a complex geometry, which leads to a real Euclidean action.

$$ I = -\frac{\beta r_+^3}{16 G_5} \Delta v V $$

• This action is the identical to the on-shell action (regulated) for the Schwarzschild-AdS black hole.

☆ The NMT/TsT does not change the leading large $N$ thermodynamic properties (follows from star product).

• Careful analysis of boundary counter-terms required to obtain the result.
Black holes and thermodynamics

Equation of state

- From the Gibbs potential easy to read off

\[ \langle E \rangle = \frac{\pi^3 T^4}{64 G_5 \mu^2} \Delta v \, V \]

\[ \langle N \rangle = P_v \frac{\Delta v}{2\pi} = \frac{\pi^2 T^4}{64 G_5 \mu^3} (\Delta v)^2 \, V \]

- This leads to an equation of state

\[ E = P \, V \]

which is the non-relativisitic conformal equation of state in 2 spatial dimensions.

- Generalizes to all dimensions easily.

Herzog, MR, Ross; Kovtun, Nickel.
Linearized fluctuations

- Study the two point function of the spatial stress tensor $\Pi_{ij}(u, x)$ to learn about $\eta$.
- Gravitational computation involves fluctuation analysis about the black hole solution.
- While generically $\delta g$, $\delta A$ and $\delta \phi$ give a coupled system: the shear mode $\delta g_{x_1 x_2}$ decouples.
- In fact $\delta g_{x_1 x_2}$ satisfies massless, minimally coupled wave equation (for zero spatial momentum).
Shear viscosity of the conformal plasma

- Remembering that the stress tensor has zero particle number $P_v = 0$, the wave equation in fact reduces to that in the Schwarzschild-AdS background, modulo

$$\omega_{\text{AdS}} = \beta \omega_{\text{Schr}}$$

- One can easily compute $\langle \Pi_{x_1 x_2} \Pi_{x_1 x_2} \rangle$ at zero spatial momentum and read off $\eta$ using a Kubo formula.

- One finds

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

- Finally, note that non-relativistic conformal invariance requires that the bulk viscosity vanish; $\zeta = 0$. 
**Non-relativistic hydrodynamics**

**Aim:** Derive the hydrodynamic equations for the non-relativistic plasma from gravity using the fluid-gravity correspondence.

### The Hard Way

- Take the asymptotically Schrödinger black hole and generalize it to a $d + 2$ parameter solution ($d$ Galilean velocities $v_i$).
- Promote $r_+$, $\beta$ and $v_i$ to fields depending on $\{u, x\}$.
- Solve bulk gravity equations order by order in derivatives of $\{u, x\}$ for asymptotically Schrödinger solutions.
- Gravity constraint equations $\rightarrow$ Navier-Stokes equations.
- Asymptotic fall-off conditions $\rightarrow$ ‘boundary’ stress tensor.
**Non-relativistic hydrodynamics**

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**The Short-Cut**

- Leading planar physics of the non-relativistic theory is the same as the parent relativistic theory.
- Obtain the stress tensor complex for the non-relativistic theory by reducing the corresponding relativistic stress tensor on the light-cone (along $v$).
- The bulk metric is obtained by TsT transformation of the asymptotically AdS fluid black hole solutions (with $\partial_v$ being the null Killing vector).
Equations for ideal relativistic hydrodynamics: These are just conservation of energy-momentum tensor and are $d + 2$ equations for $d + 2$ variables (fluids on $\mathbb{R}^{d+1,1}$)

$$\nabla_\mu T^{\mu\nu} = 0.$$

$$T^{\mu\nu} = (\epsilon_{\text{rel}} + P_{\text{rel}}) u^\mu u^\nu + P_{\text{rel}} \eta^{\mu\nu},$$
Relativistic & non-relativistic hydrodynamics

Equations for ideal non-relativistic hydrodynamics: These are again conservation equations:

- Continuity equation: \( \partial_t \rho + \partial_i (\rho v^i) = 0 \),
- Momentum conservation: \( \partial_t (\rho v^i) + \partial_j \Pi^{ij} = 0 \),
- Energy conservation: \( \partial_t \left( \varepsilon + \frac{1}{2} \rho v^2 \right) + \partial_i j^i_\varepsilon = 0 \),

where we have defined

- spatial stress tensor: \( \Pi^{ij} = \rho v^i v^j + \delta^{ij} P \)
- energy flux: \( j^i_\varepsilon = \frac{1}{2} (\varepsilon + P) v^2 v^i \)
Light-cone reduction of ideal relativistic hydrodynamics

Consider the relativistic stress tensor in light-cone coordinates $x^\pm = \{u, v\}$.

\[ \partial_+ T^{++} + \partial_i T^{+i} = 0, \quad \partial_+ T^{+i} + \partial_j T^{ij} = 0, \quad \partial_+ T^{+-} + \partial_i T^{-i} = 0, \]

which allows us to identify

\[ T^{++} = \rho, \quad T^{+i} = \rho v^i, \quad T^{ij} = \Pi^{ij}, \]
\[ T^{+-} = \varepsilon + \frac{1}{2} \rho v^2, \quad T^{-i} = j^i. \]
Light-cone reduction of ideal relativistic hydrodynamics

The map between relativistic and non-relativistic variables:

\[ u^+ = \sqrt{\frac{1}{2} \frac{\rho}{\varepsilon + P}}, \quad u^i = u^+ v^i, \]

\[ P_{\text{rel}} = P, \quad \varepsilon_{\text{rel}} = 2\varepsilon + P. \]

The component of the relativistic velocity \( u^- \) can be determined using the normalization condition \( u_\mu u^\mu = -1 \) to be

\[ u^- = \frac{1}{2} \left( \frac{1}{u^+} + u^+ v^2 \right). \]
The map can be extended to incorporate dissipative effects.

The conformal relativistic stress tensor at first order reads:

\[ T^\mu{}\nu = (\epsilon_{\text{rel}} + P_{\text{rel}}) u^\mu u^\nu + \eta^{\mu\nu} P_{\text{rel}} - 2 \eta_{\text{rel}} \tau^{\mu\nu} \]

with \( \tau^{\mu\nu} \) being the shear tensor.

Light-cone reduction is as before, with derivative corrections to the map between velocities.

Can use the map to derive the non-relativistic transport coefficients at first order.
Non-relativistic transport coefficients:

- We find for the shear viscosity

\[ \eta_{\text{rel}} = \frac{\eta}{u^+}. \]

- The heat conductivity is given by

\[ \kappa = 2\eta \frac{\varepsilon + P}{\rho T}. \]

- The dimensionless ratio Prandtl number defined as the ratio of kinematic viscosity \( \nu \) to thermal diffusivity \( \chi \) is 1.

\[ \text{Pr} = \frac{\nu}{\chi}, \quad \nu = \frac{\eta}{\rho}, \quad \chi = \frac{\kappa}{\rho c_p}. \]
Salient points

- Holographic dual for system with Galilean conformal invariance, using D-brane construction.
- D-branes probing a Null Melvin geometry naturally give rise to such non-relativistic CFTs.
- Discussed thermodynamics and some hydrodynamic properties of such plasmas.
- As usual, brane engineering leads to systems where $\eta/s$ takes on the universal value $1/4\pi$.
- Can discuss conformal non-relativistic hydrodynamics for the system: derived transport coefficients at first order and constructed dual gravity solutions.
Moral from cold atoms

Non-relativistic symmetry and non-distinguishability

• The fact that the dual theory has non-relativistic invariance, necessitates that the bulk spacetime be non-distinguishing.

• Otherwise it would not be possible for a sensible bulk spacetime with local Lorentz invariance to be dual to a theory with Galilean symmetry.

• These theories provide a playground to explore interesting physics in strongly coupled non-relativistic CFTs, e.g. thermodynamics, transport coefficients, etc..

• At the same time they also provide important lesson for Quantum Gravity, viz., non-distinguishingness is ‘acceptable’.
Open questions

• Understand what asymptotically Schrödinger spacetime means.
  ★ Define the precise fall-off conditions for asymptopia.
  ★ Identify the conserved charges unambiguously.

• Generalizations to study non-relativistic CFTs on compact manifolds.
  - Yamada

• What classes of critical exponents $\nu$ are accessible in string theory?
  ★ $\nu = 2$ encountered in lightlike NCYM.
  ★ $\nu = 3$ seen for a particular 5-form deformation of $\text{AdS}_5 \times S^5$.
    - Maldacena, Martelli, Tachikawa
  ★ Half-integral $\nu$ established via a different solution generating technique.
    - Hartnoll, Yoshida