Accurate bottom quark mass from sum rules for decay constants of B – mesons

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We show that Borel QCD sum rules for heavy–light currents yield very strong correlations between the b-quark mass m_b and the B-meson decay constant f_B:

$$\delta f_B/f_B \approx -8\,\delta m_b/m_b.$$

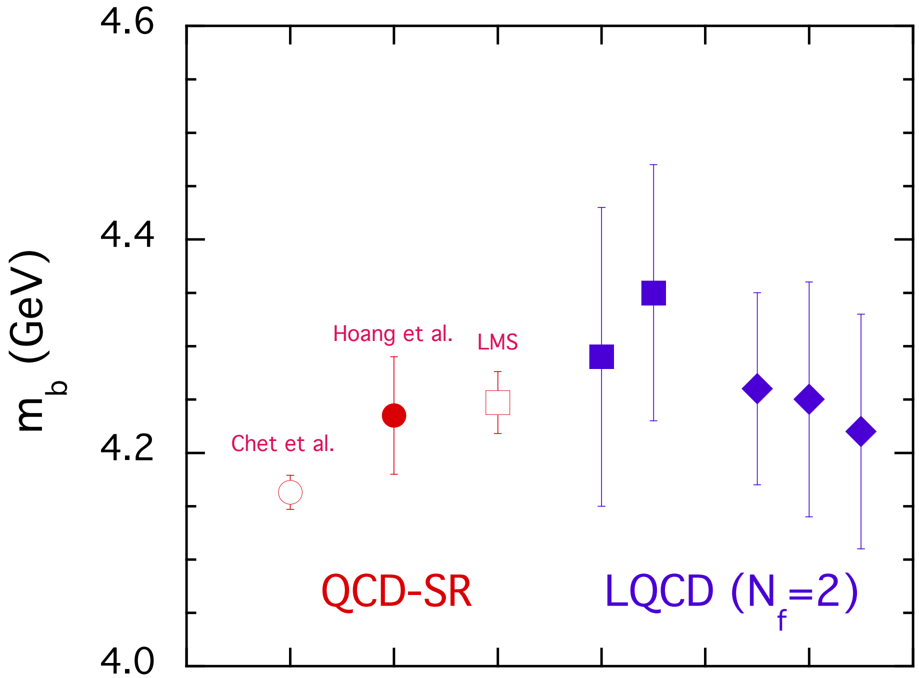
This fact opens the possibility of an accurate extraction of m_b from QCD sum rules using f_B as input. Combining precise lattice QCD determinations of f_B with our sum-rule analysis based on heavy–light correlation function leads to

$$\overline{m}_b(\overline{m}_b) = (4.247 \pm 0.034)\ \textbf{GeV}$$

For the ratio vector/pseudoscalar decay constants we obtain $f_{B*}/f_B = 0.97 \pm 0.06$.

Based on W.Lucha, D.Melikhov, S.Simula, "Accurate bottom-quark mass from Borel QCD sum rules for f_B and f_{B_s}", Phys. Rev. **D88***, 056011 (2013); "Decay Constants of Beauty Mesons from QCD Sum Rules", arXiv:1410.6684.*

Precise knowledge of the b-quark mass is highly desirable [$m_b(\mu) \equiv \overline{m}_b(\mu)$, $m_b \equiv \overline{m}_b(\overline{m}_b)$]



Moment SRs for $\bar{b}b$ two-point functions in pQCD with 4-loop accuracy vs experimental data:

low-n moments (Chetyrkin et al): $m_b = 4.163 \pm 0.016$ GeV

large-n moments (Hoang et al): $m_b = 4.235 \pm 0.055_{\rm (pert)} \pm 0.03_{\rm (exp)}$ GeV

We report that Borel QCD sum rules for heavy–light correlators provide the possibility to extract m_b with comparable accuracy if a precise value for f_B is used as input.

Correlation between f_B and m_b

Explore the sensitivity of f_B to the precise value of the b-quark mass:

In a nonrelativistic potential model the following relationship between the ground-state wave function at the origin, $\psi(r = 0)$, and the ground-state binding energy ε:

$$|\psi(r = 0)| \propto \varepsilon^{3/2}.$$

The decay constant $f_B \sim \psi(r = 0)$; in the heavy-quark limit $f_B \propto 1/\sqrt{m_Q}$:

$$\sqrt{M_B}\, f_B = \kappa\, (M_B - m_Q)^{3/2}$$

Dependence of f_B on small variations δm_Q of the heavy-quark (pole) mass near some m_Q (keeping M_B fixed!):

$M_B = 5.27$ GeV**;** $f_B \approx 200$ MeV **for** $m_Q \approx 4.6 \div 4.7$ GeV

$\rightarrow \kappa \approx 0.9 \div 1.0$ **and** $\delta f_B \approx -0.5\, \delta m_Q$**.**

The sensitivity of f_B to the precise value of the heavy-quark mass should be very high!

$$\frac{\delta f_B}{f_B} \approx -(11 \div 12)\, \frac{\delta m_Q}{m_Q}.$$

Recent QCD sum-rule results for f_B based on 3-loop two-point function have been presented:

	Narison'2001	**Jamin'2002**	**LMS'2009**	**Narison'2012**
m_b (GeV)	4.05 ± 0.06	4.21 ± 0.05	4.245 ± 0.025	4.236 ± 0.069
f_B (MeV)	203 ± 23	210 ± 19	193 ± 15	206 ± 7

Not all the results here are equally trustable.

Recall: the values of the ground-state parameters are strongly influenced by

- **the reliable perturbative expansion of the two-point function;**
- **the way of fixing the auxiliary parameters of the sum-rule approach, particularly, the effective continuum threshold.**

For fixed inputs of the correlator (condensates, α_s, etc.) we find

$$f_B(m_b) = \left(192.0 - 37\,\frac{m_b - 4.247\ \textbf{GeV}}{0.1\ \text{GeV}} \pm 4_{(\text{syst})}\right)\text{MeV}.$$

Strong correlation between f_B and m_b!

Combining our sum-rule analysis with f_B from lattice QCD $f_B^{\text{LQCD}} = (191.5 \pm 7.3)$ MeV **leads to**

$$m_b = (4.247 \pm 0.027_{(\text{OPE}+\text{f}_\text{B})} \pm 0.011_{(\text{syst})}\ \text{GeV}.$$

Correlation function, OPE, and heavy – quark mass

The basic object is T-product of 2 pseudoscalar currents, $j_5(x) = (m_b + m)\,\bar{q}(x) i\gamma_5 b(x)$,

$$\Pi(p^2) = i \int d^4x\, e^{ipx} \left\langle 0 \left| T\left(j_5(x) j_5^\dagger(0)\right) \right| 0 \right\rangle$$

and its Borel image: $\Pi(p^2) = \int \frac{ds}{s-p^2}\rho(s) \to \Pi(\tau) = \int ds \exp(-s\tau)\rho(s)$

- **OPE for $\Pi(\tau)$ in QCD:**

$$\Pi(\tau) = \int_{(m_b+m)^2}^{\infty} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu).$$

$$\Pi_{\rm power}(\tau, \mu = m_Q) = (m_Q + m)^2 e^{-m_Q^2\tau}\left\{-m_Q\langle\bar{q}q\rangle\left[1 + \frac{2C_F\alpha_s}{\pi}\left(1 - \frac{m_Q^2\tau}{2}\right) + \dots\right] + \frac{1}{12}\left\langle\frac{\alpha_s}{\pi}GG\right\rangle\right\}.$$

- **Using hadron intermediate states:**

$$\Pi(\tau) = f_B^2 M_B^4 e^{-M_B^2\tau} + \int_{s_{\rm phys}}^{\infty} ds\, e^{-s\tau}\rho_{\rm hadr}(s)$$

Here $s_{\rm phys} = (M_{B^*} + M_P)^2$, **and** f_B **is the decay constant defined by**

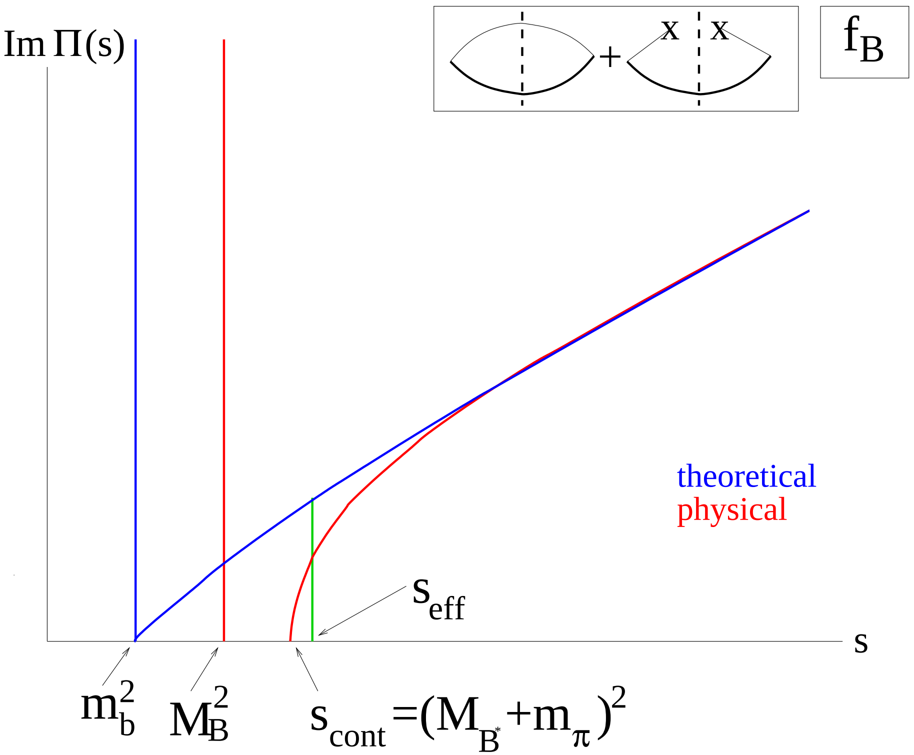
$$(m_b + m)\langle 0|\bar{q}i\gamma_5 b|B\rangle = f_B M_B^2.$$

Sum rule is obtained by equating these two representations:

$$\Pi(\tau) = f_B^2 M_B^4 e^{-M_B^2\tau} + \int_{s_{\rm phys}}^{\infty} ds\, e^{-s\tau}\rho_{\rm hadr}(s) = \int_{(m_b+m)^2}^{\infty} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu).$$

To exclude the excited-state contributions, one adopts the *duality Ansatz***:**

Contributions of the excited states are counterbalanced by the perturbative contribution above an *effective continuum threshold*, $s_{\rm eff}(\tau)$ **which differs from the physical continuum threshold.**



Applying the duality assumption yields:

$$f_B^2 M_B^4 e^{-M_B^2\tau} = \int\limits_{(m_b+m)^2}^{s_{\rm eff}(\tau)} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu) \equiv \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).$$

The rhs is the *dual correlator* $\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau))$**.**

Even if the QCD inputs $\rho_{\rm pert}(s,\mu)$ **and** $\Pi_{\rm power}(\tau,\mu)$ **are known, the extraction of the decay constant requires, in addition, a criterion for determining** $s_{\rm eff}(\tau)$**.**

As first step, we need a reasonably convergent OPE for both correlator and dual correlator.

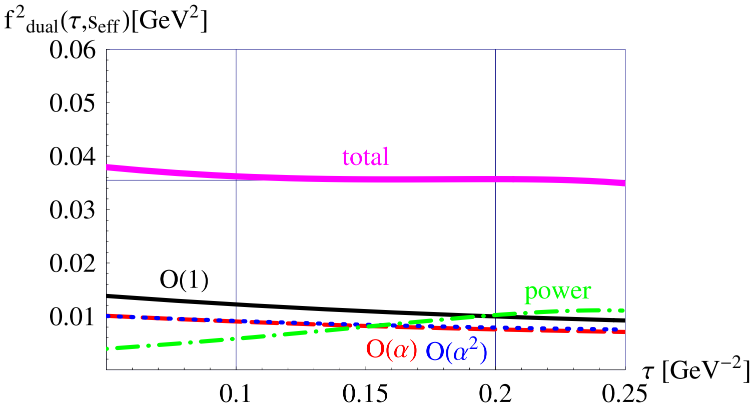
The best-known 3-loop calculations of the perturbative spectral density have been performed in form of an expansion in terms of the $\overline{\rm MS}$ **strong coupling** $\alpha_{\rm s}(\mu)$ **and the pole mass** M_b**:**

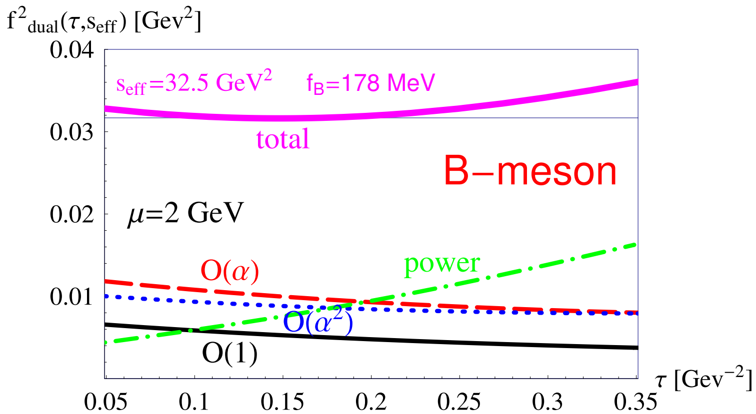
$$\rho_{\rm pert}(s,\mu) = \rho^{(0)}(s, M_b^2) + \frac{\alpha_{\rm s}(\mu)}{\pi}\rho^{(1)}(s, M_b^2) + \left(\frac{\alpha_{\rm s}(\mu)}{\pi}\right)^2 \rho^{(2)}(s, M_b^2, \mu) + \cdots.$$

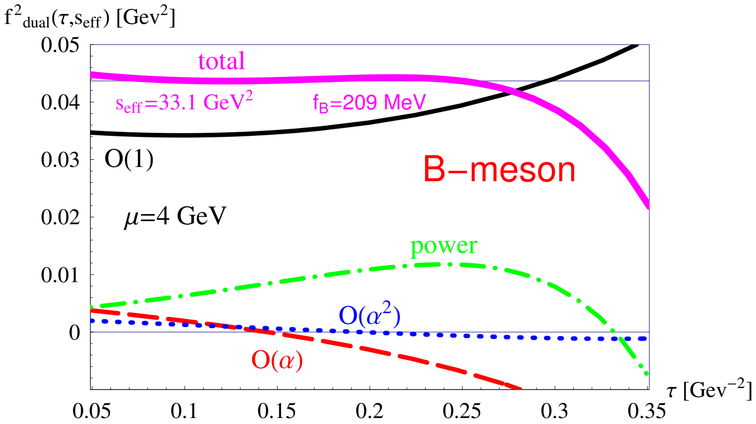
An alternative option is to reorganize the perturbative expansion in terms of the running $\overline{\rm MS}$ **mass,** $\overline{m}_b(\nu)$**, by substituting** M_b **in the spectral densities** $\rho^{(i)}(s, M_b^2)$ **via its perturbative expansion in terms of the running mass** $\overline{m}_b(\nu)$

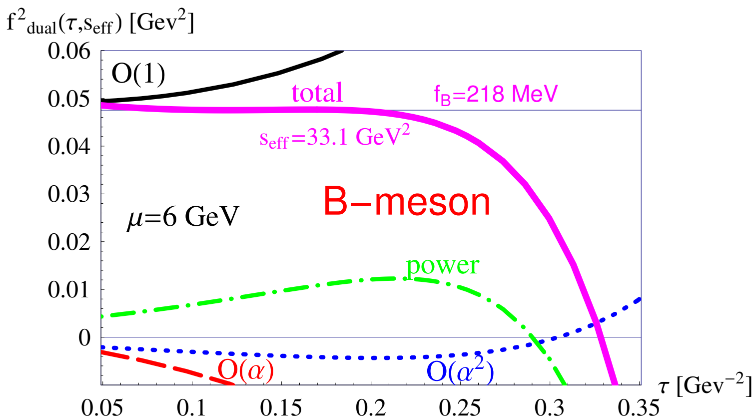
$$M_b = \overline{m}_b(\nu)\left(1 + \frac{\alpha_s(\nu)}{\pi} r_1 + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2 r_2 + \ldots\right).$$

OPE in terms of *b*-quark pole and $\overline{\mathrm{MS}}$ mass for $\overline{m}_b(\overline{m}_b) = 4.18$ GeV (PDG):









Lessons:

1. **For dual correlator through the heavy-quark pole mass, no convergence of the perturbative expansion;** *In pole-mass scheme one cannot expect higher orders to give smaller contributions*.
2. **In terms of the heavy-quark $\overline{\mathrm{MS}}$ mass the hierarchy depends on the scale μ. One can should work in the μ-range where the hierarchy is "good".**

Extraction of the decay constant

1. *The Borel-parameter τ window*

is chosen such that (i) the OPE gives an accurate description of the exact correlator (i.e., all higher-order radiative and power corrections are under control) and (ii) the ground state gives a "sizable" contribution to the correlator. Our τ-window for the $B_{(s)}$ mesons is $0.05 \lesssim \tau\,(\mathrm{GeV}^{-2}) \lesssim 0.175$.

2. *Algorithm for the effective continuum threshold*

To find $s_{\rm eff}(\tau)$, we employ a previously developed algorithm which provides a reliable extraction of the ground-state parameters in quantum-mechanics and of the charmed-meson decay constants in QCD. We introduce the *dual invariant mass* $M_{\rm dual}$ and the *dual decay constant* $f_{\rm dual}$

$$M_{\rm dual}^2(\tau) \equiv -\frac{d}{d\tau}\log\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)), \qquad f_{\rm dual}^2(\tau) \equiv M_B^{-4}\, e^{M_B^2\tau}\,\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).$$

The dual mass should reproduce the true ground-state mass M_B; the deviation of $M_{\rm dual}$ from M_B measures the contamination of the dual correlator by excited states. Starting from an Ansatz for $s_{\rm eff}(\tau)$ and requiring a minimum deviation of $M_{\rm dual}$ from M_B in the τ-window generates a variational solution for $s_{\rm eff}(\tau)$. We consider polynomials in τ, including also a τ-independent constant:

$$s_{\rm eff}^{(n)}(\tau) = \sum_{j=0}^{n} s_j^{(n)}\tau^j.$$

We obtain $s_j^{(n)}$ by minimizing the squared difference between $M_{\rm dual}^2$ and M_B^2 in the τ-window:

$$\chi^2 \equiv \frac{1}{N}\sum_{i=1}^{N}\left[M_{\rm dual}^2(\tau_i) - M_B^2\right]^2.$$

Uncertainties in the extracted decay constant

The resulting f_B is sensitive to the input values of the OPE parameters — which determines what we call the *OPE-related error* — and to the details of the adopted prescription for fixing the behaviour of the effective continuum threshold $s_{\rm eff}(\tau)$ — the *systematic error*.

OPE – related error

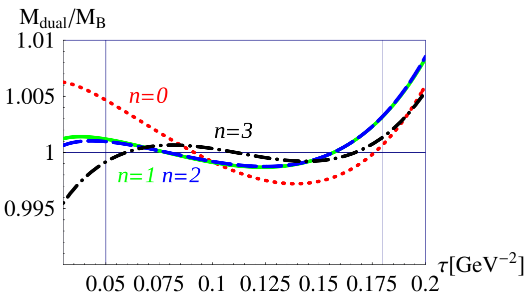
We estimate the size of the OPE-related error by perform a bootstrap analysis, assuming Gaussian distributions for all OPE parameters but the renormalization scales. For the latter, we assume uniform distributions in the range $3 \le \mu, \nu\ ({\rm GeV}) \le 6$. The resulting distribution of the decay constant turns out to be close to Gaussian shape. Hence, the quoted OPE-related error is a Gaussian error.

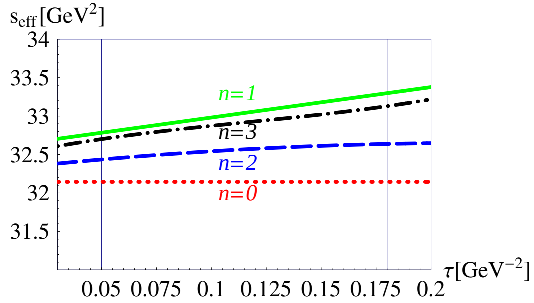
Systematic error

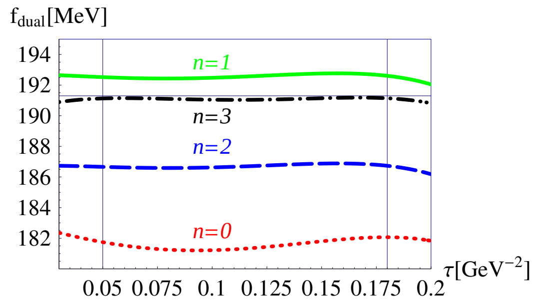
The systematic error, related to the limited intrinsic accuracy of the method of sum rules, is a subtle point. In quantum mechanics, we observed that considering polynomial parameterizations of the effective continuum threshold $s_{\rm eff}(\tau)$, the band of results obtained from linear, quadratic, and cubic Ansätze for $s_{\rm eff}(\tau)$, encompasses the true value of the decay constant. Thus, the half-width of this band may be regarded as a realistic estimate for the systematic uncertainty of the prediction.

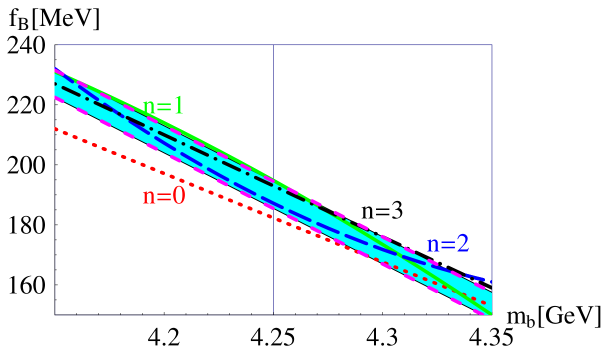
Decay constant of B – meson

For $m_b \equiv \overline{m}_b(\overline{m}_b) = 4.247$ GeV, $\mu = \nu = m_b$, and central values of the other relevant parameters:









Our results for f_B may be parameterized by (for fixed values of other OPE parameters)

$$f_B^{\text{dual}}(m_b, \mu = \nu = m_b, \langle\bar{q}q\rangle) = \left[192.0 - 3.7\left(\frac{m_b - 4.247\ \textbf{GeV}}{10\ \textbf{MeV}}\right) + 4\left(\frac{|\langle\bar{q}q\rangle|^{1/3} - 269\ \textbf{MeV}}{10\ \textbf{MeV}}\right) \pm 4_{(\text{syst})}\right]\textbf{MeV},$$

Performing the bootstrap analysis we find

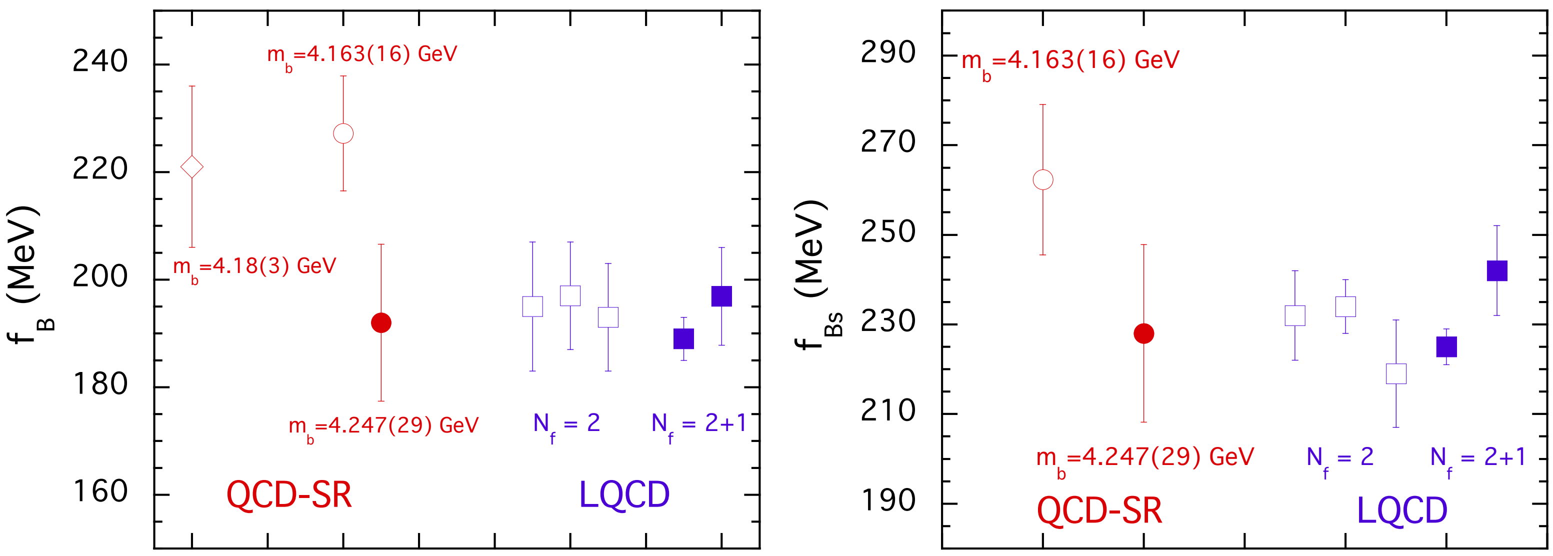
$$f_B = \left(192.0 \pm 14.3_{\rm (OPE)} \pm 4_{\rm (syst)}\right) \textbf{MeV}.$$

A similar procedure yields for the B_s meson

$$f_{B_s} = \left(228.0 \pm 19.4_{\rm (OPE)} \pm 4_{\rm (syst)}\right) \textbf{MeV}.$$

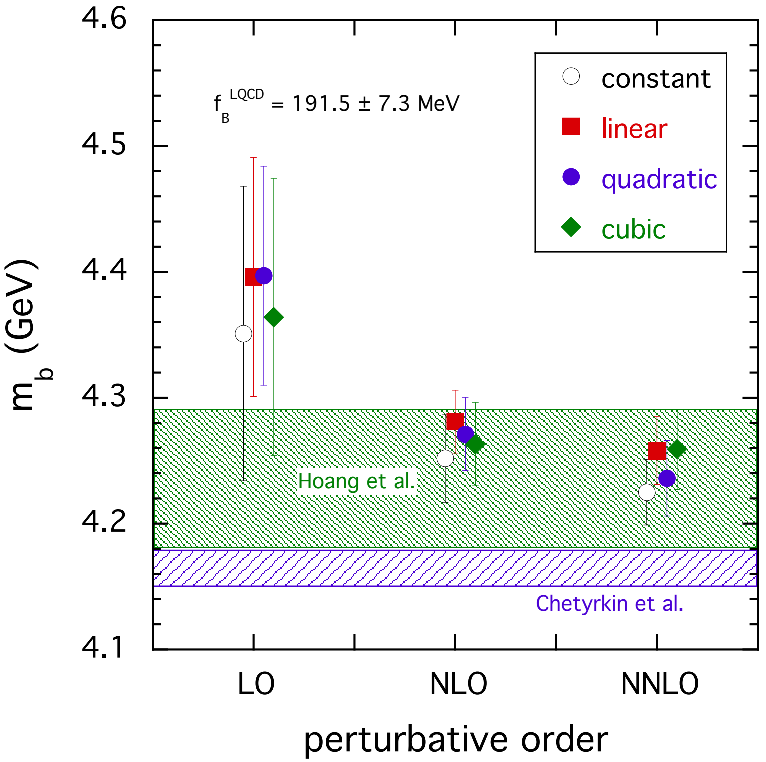
and

$$f_{B_s}/f_B = 1.184 \pm 0.023_{\rm (OPE)} \pm 0.007_{\rm (syst)}.$$



Extraction of the bottom – quark mass

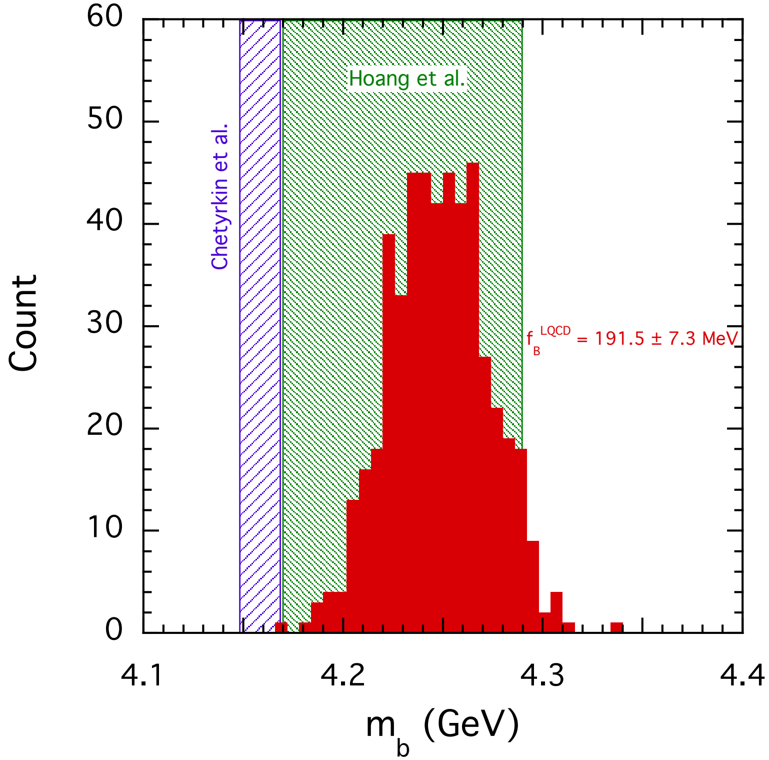
Using lattice average $f_B = 191.5$ MeV and applying our algorithms yields:



$$m_b^{LO} = (4.38 \pm 0.1 \pm 0.02_{syst})\ \text{GeV}$$
$$m_b^{NLO} = (4.27 \pm 0.04 \pm 0.015_{syst})\ \text{GeV}$$
$$m_b^{NNLO} = (4.247 \pm 0.027 \pm 0.011_{syst})\ \text{GeV}$$

Moving from LO to NLO of the perturbative expansion (i) decreases sizeably m_b and reduces OPE-error. The extracted values of m_b exhibit a nice "convergence" depending on the accuracy of the perturbative correlation function.

The N^3LO correction is not known. Nevertheless, we do not expect a sizeable shift of the central value of m_b, but expect a reduction of the OPE-error.



Distribution of m_b as obtained by the bootstrap analysis:

- **Gaussian distributions for**

$f_B = 191.5 \pm 7.3\,\text{MeV}$

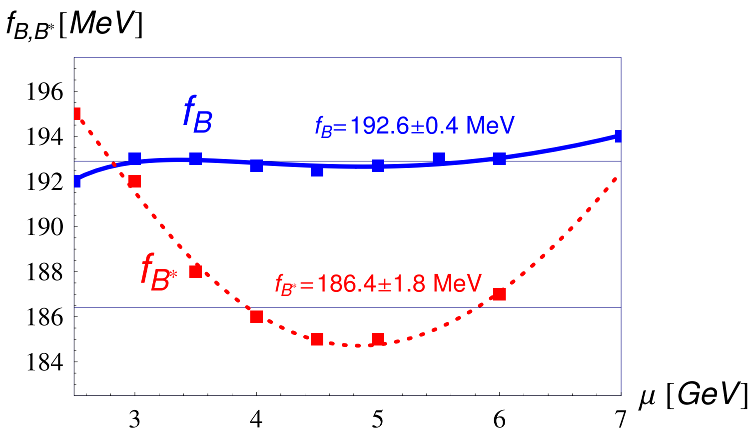
and OPE parameters

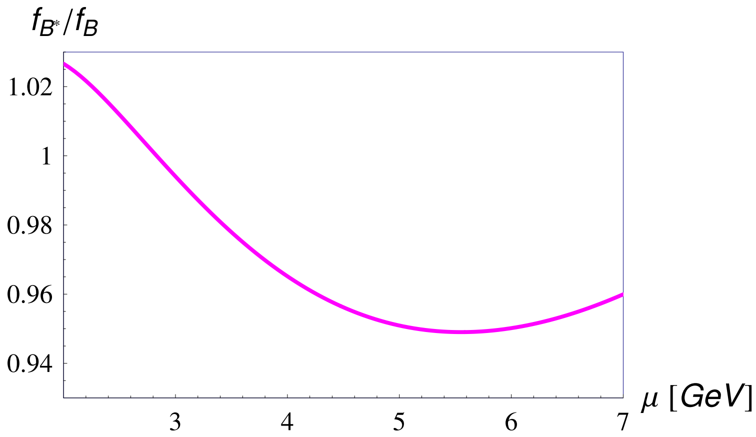
$\alpha_s(M_Z) = 0.1184 \pm 0.0007,\; m_d(2\,\text{GeV}) = 3.5 \pm 0.5\,\text{MeV},\; m_s(2\,\text{GeV}) = 95 \pm 5\,\text{MeV}$

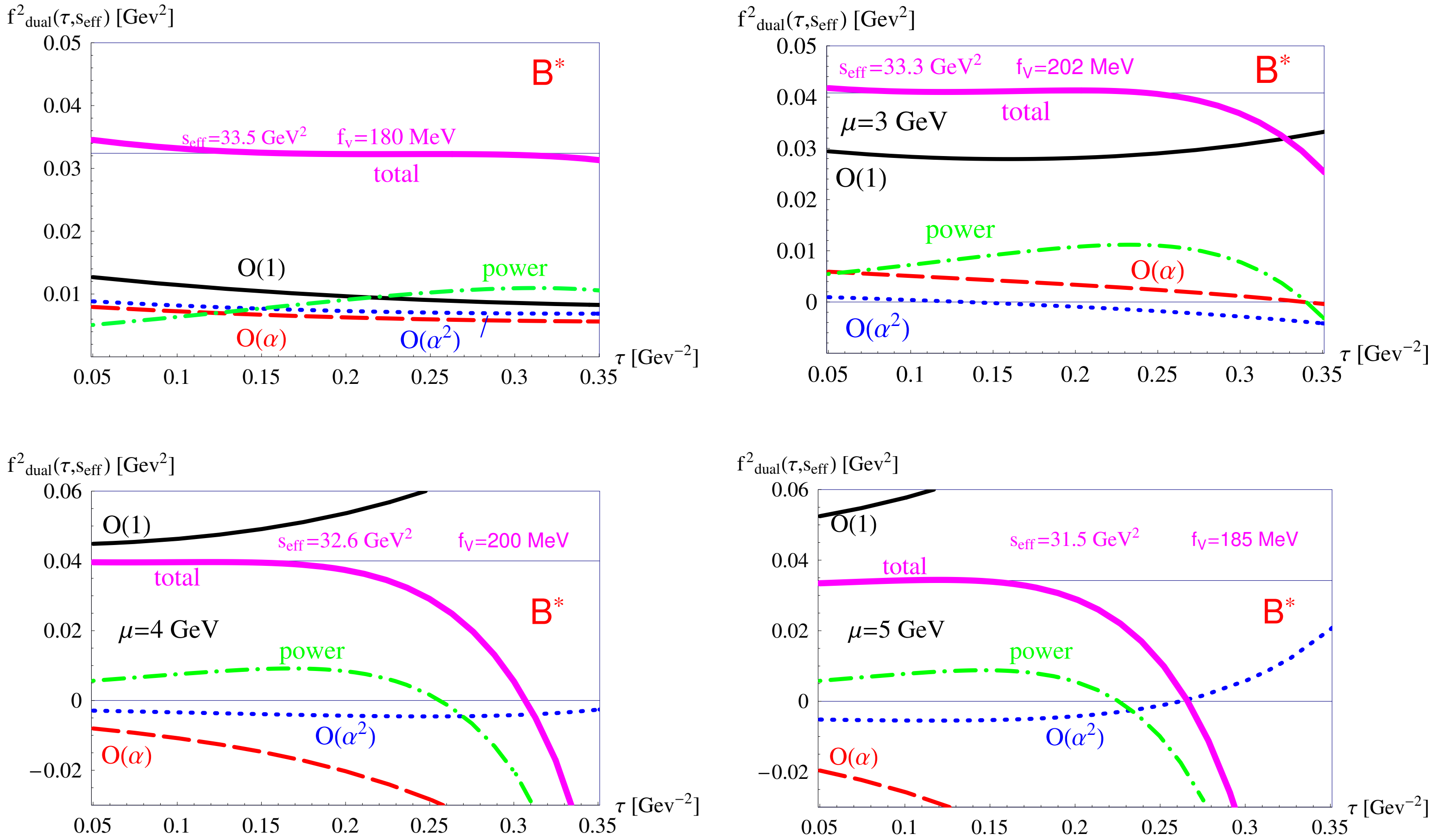
$<\alpha_s/\pi\, GG> = 0.024 \pm 0.012\,\text{GeV}^4,\; <\bar{q}q>(2\,\text{GeV}) = -(269 \pm 17\,\text{MeV})^3$

- **Uniform distributions for the renormalization scales μ and ν in the range**

$3\,\text{GeV} < \mu, \nu < 6\,\text{GeV}$


f_{B,B^*} [MeV]
f_B
f_B=192.6±0.4 MeV
f_{B^*}
f_{B^*}=186.4±1.8 MeV
μ [GeV]


f_{B^*}/f_B
μ [GeV]



f²dual(τ,seff) [Gev²]
B*
seff=33.5 GeV²
fv=180 MeV
total
O(1)
power
O(α)
O(α²)
τ [Gev⁻²]
f²dual(τ,seff) [Gev²]
seff=33.3 GeV²
fV=202 MeV
B*
μ=3 GeV
total
O(1)
power
O(α)
O(α²)
τ [Gev⁻²]
f²dual(τ,seff) [Gev²]
O(1)
seff=32.6 GeV²
fV=200 MeV
total
B*
μ=4 GeV
power
O(α²)
O(α)
τ [Gev⁻²]
f²dual(τ,seff) [Gev²]
O(1)
seff=31.5 GeV²
fV=185 MeV
total
B*
μ=5 GeV
power
O(α²)
O(α)
τ [Gev⁻²]

Summary

We presented a detailed QCD sum-rule analysis of the B- and B_s-meson decay constants, with particular emphasis on the study of the errors in the extracted decay-constant values: the OPE uncertainty due to the errors of the QCD parameters and the intrinsic error of the sum-rule approach due to the limited accuracy of the extraction procedure. Our main findings are:

- **The extraction of hadronic properties is improved by allowing a Borel-parameter dependence for the effective continuum threshold, which increases the accuracy of the duality approximation. Considering suitably optimized polynomial Ansätze for the effective continuum threshold provides an estimate of the intrinsic uncertainty of the method of QCD sum rules.**
- **For beauty mesons, a strong correlation between m_b and the sum-rule result for f_B is reported:**

$$\frac{\delta f_B}{f_B} \approx -8\,\frac{\delta m_b}{m_b}.$$

Combining our sum-rule analysis with the latest results for f_B and f_{B_s} from lattice QCD yields

$$m_b = 4.247 \pm 0.027_{(OPE)} \pm 0.018_{(exp)} \pm 0.011_{syst}\ \text{GeV}$$

OPE error:

14 MeV (μ, ν), 20 MeV (quark cond.), 7 MeV (gluon cond.), 8 MeV (α_s), 4 MeV (light-quark mass).

Good news is that the systematic uncertainty is relatively small.

Our value of m_b is extracted from the heavy–light correlator known to $O(\alpha_s^2)$ accuracy. Since the value of m_b is changing only marginally when moving from the $O(\alpha_s)$ to $O(\alpha_s^2)$ accuracy of the correlator, we do not expect the inclusion of the presently unknown $O(\alpha_s^3)$ correction to lead to a substantial change in the extracted value of m_b.

Our result compares well with

$m_b = (4.209 \pm 0.050)$ **GeV**

found from moment sum rules for heavy–heavy correlators evaluated to the same $O(\alpha_s^2)$ accuracy as in our analysis.

We observe an excellent agreement with the prediction of the Υ sum rule

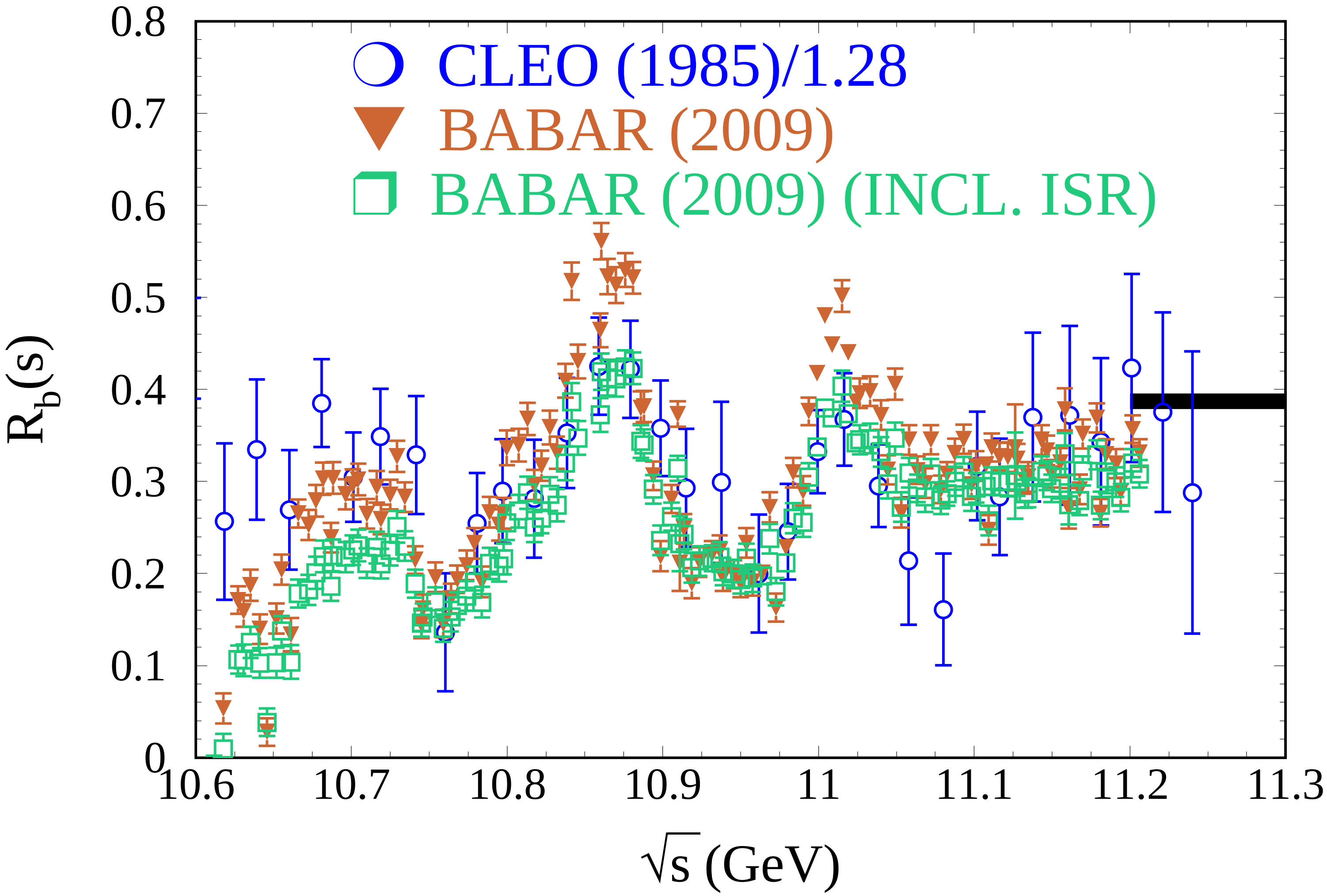
$m_b = (4.235 \pm 0.055_{\rm (pert)} \pm 0.003_{\rm (exp)})$ **GeV**.

We see, however, a pronounced tension with the prediction

$m_b = (4.163 \pm 0.016)$ **GeV**,

based on moment sum rules for heavy–heavy correlators calculated to $O(\alpha_s^3)$ accuracy. The origin of this disagreement requires further considerations.

Properly formulated Borel QCD sum rules for heavy–light correlators provide a competitive tool for the reliable calculation of heavy-meson properties and for the extraction of basic QCD parameters by making use of the results from lattice QCD and/or the experimental data.


CLEO (1985)/1.28
BABAR (2009)
BABAR (2009) (INCL. ISR)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
R_b(s)
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3
√s (GeV)